Niels Henrik Abel

The quadratic formula, which gives us the roots of any degree-2 polynomial from its coefficients, is among the best-known theorems in mathematics, with a 4000-year history. In the early 1500s, del Ferro and Tartaglia, independently, opened a new era in math by inventing the cubic formula that solves the corresponding problem in degree 3, and Ferrari had finished off the degree-4 problem by 1540. It was natural to expect progress to continue, but by the time Niels Henrik Abel was born in 1802, there was still no quintic formula.

Abel's childhood was very difficult. His father was a politician who supported Norwegian independence (from first Denmark, then Sweden). Abel was home-schooled, taught by his father, but this was a bad situation: the father would eventually lose his seat in the legislature for making false accusations against other members and for drinking too much. In 1815, with his family in desperate financial trouble, Abel was sent to a bad boarding school: the math teacher was fired for beating a student to death!

As a silver lining to that tragedy, Abel got a new math teacher who was excellent. He recognized Abel's ability and introduced him to more advanced topics, including active research in mathematics. He also got Abel a scholarship at the University of Christiania (now Oslo), and collected enough donations that Abel was actually able to enroll. Before graduating (in two years), Abel thought he had discovered the long-sought quintic formula; sadly, when he had to use it to do an example, he realized he'd made a mistake.

At the University, Abel again met a teacher who supported him financially, which gave him the freedom to work on math. In these years, he was the first to solve an integral equation (think of a differential equation, but backwards). More importantly, in 1823, he solved the problem of quintic polynomials in a way that absolutely no one had expected: It was impossible to solve degree-5 polynomial equations using only arithmetic operations and radicals. Not only that, but there was a very surprising reason: There are specific polynomials, such as $x^5 - x - 1$, whose roots can't be written down at all in terms of radicals. He also showed that a polynomial would be solvable by radicals when a related group's operation was commutative; such groups are now called "abelian."

The next year, with help from his school and the Norwegian government, he traveled to France and Germany to talk with the important mathematicians there, but the trip was discouraging: Cauchy barely looked at Abel's work, Gauss didn't read it at all, and Abel could only afford one meal a day. Back at home, he continued to do good work, but he became seriously ill the next year, while traveling by sled to visit his fiancée for Christmas, and died a few months later at the age of 26.

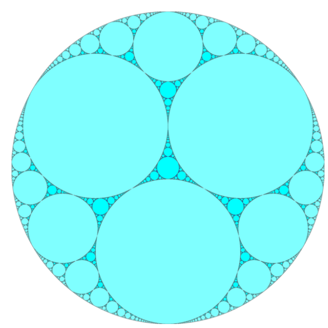
Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Abel/`

Apollonius

Apollonius of Perga lived in the ancient Greek empire, though he was born in what is now Turkey and died in the city of Alexandria, which is now in Egypt. Alexandria had been the home of Euclid, and the site of the famous Library; Apollonius went there to learn from Euclid's students, and then became a teacher at the University.

Apollonius is famous as the author of *Conics*, an eight-book study of the conic sections (cross-sections of a cone). It is the oldest surviving work on the subject, and the original source of the words "ellipse," "parabola," and "hyperbola" as the names of these curves. Unlike any previous mathematician, Apollonius could work with any cone formed by lines through a given point and circle. Thus his cones were double cones, infinitely tall, which might be oblique (with circular cross-sections not perpendicular to the axis through their centers). All three of these properties represented major advances beyond what had been done before. Specifically, since Apollonius was the first to use double cones, he was the first to regard hyperbolas as having two branches.

Given any segment $\overline{AB}$, one can form the family of curves on which the ratio of the distances to A and to B remains constant. It was Apollonius who first showed that the curves in this family really are circles, and that each of these "Apollonian circles" will intersect at right angles with any circle that passes through both A and B.



The eighth book of *Conics* is entirely lost to time, and only the first four still exist in the original Greek. The fifth, sixth, and seventh books were preserved in Arabic translations, created and used by mathematicians in the Golden Age of Islam. Nothing else that Apollonius wrote has survived, but we know that he wrote many other works, because they're cited by other authors. Pappus, who lived about 500 years after Apollonius, refers to six other works of Apollonius, but only one (*Cutting of a ratio*) has come down to the present day. Another of these works, *Tangencies*, introduced the "problem of Apollonius" — the construction of a circle tangent to three given circles. Almost two thousand years later, Leibniz described how, by repeatedly constructing smaller and smaller tangent circles, one could generate a beautiful fractal which we now call an "Apollonian gasket" (right).

Images:

- MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Apollonius/`
- Roice3, `https://commons.wikimedia.org/wiki/File:H3_i33_UHS_plane_at_infinity.png`

Jacob Bernoulli

Jacob Bernoulli was the son of influential parents, who made him study theology. Against their wishes, he worked on astronomy at the same time, and got a job as a lecturer at his hometown university. At about this time, he was able to show that the value of an investment earning compound interest approaches a finite limit as the number of compounding intervals approaches infinity, and in doing so, he was the first to identify the constant $e = 2.71828\ldots$, though almost 50 years would pass before Euler gave it the name e.

At the time, Christiaan Huygens had recently solved a major research problem. The goal was to find the "tautochrone" — a curve for which a particle sliding down the curve, starting from rest, would take the same amount of time to reach the bottom regardless of its starting position on the curve. Huygens showed that the cycloid (the path traced out by a point on a wheel rolling on a horizontal surface) has this property. Bernoulli found another proof of the same fact by modeling the problem with a differential equation. In this paper, in 1690, Bernoulli was the first to use the word "integral" in a calculus context. Five years later, he posed what is now called the Bernoulli equation, $y' = p(x)y + q(x)y^n$, as a challenge to the mathematical community, and received solutions both from Leibniz and from his own brother Johann (of whom more later).

Jacob Bernoulli began the mathematical study of probability in the revolutionary book *Ars Conjectandi* (The Art of Conjecturing). Published after his death, it includes his proof of the "weak law of large numbers," shows how to count permutations and combinations, and introduces the modern definition of expected value, the Bernoulli distribution (for random variables whose only values are 0 and 1), and the binomial distribution (for the number of 1s occurring in n independent Bernoulli trials). In the process, he showed how to find sums of powers of consecutive integers, such as $1^{10} + 2^{10} + \cdots + 1000^{10}$, with the help of a sequence we now call the Bernoulli numbers: $\{B_n\} = \left\{1, \frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, \ldots\right\}$.

In addition to Jacob, the Bernoulli family produced seven more mathematicians in three generations. The best of these were Jacob's younger brother Johann and Johann's son Daniel. To add to the confusion, Switzerland was (and is) a multilingual country, and Jacob used the name Jacques among French speakers and James in English, while Johann went by Jean and John respectively. It was Johann who first proved l'Hôpital's rule, in a calculus textbook that the Marquis de ł'Hôpital put own his name on. Jacob was at first Johann's math tutor, but the competitive men became bitter adversaries, with Jacob declaring publicly (and falsely) that Johann only knew how to repeat what he'd heard from Jacob. It was Daniel who studied the behavior of moving fluids, including the "Bernoulli principle" that a faster air flow must correspond to a decrease in pressure.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Bernoulli/`

George Boole was the son of a shoemaker. The family was poor, and Boole wasn't able to go to high-quality schools. He had to teach himself much of what he learned, including several languages. To bring in money, he took a teaching job at 16, and opened his own school when he was 19. As a teacher, he began to study mathematics on his own, learning calculus entirely from a book, and proceeding to study the research of active mathematicians.

George Boole

Gradually, Boole mastered the new ideas that developed into the field of abstract algebra. By showing how these new tools could be useful in solving differential equations, he built a reputation as a creative thinker and an important mathematician, and earned a job as a professor in Ireland.

When he was 37, Boole started explaining calculus to 20-year-old Mary Everest (Mount Everest is named after her uncle). They married three years later, and she ultimately became a writer (on the topics of math, education, and spiritualism), a university librarian, and a private math tutor, thus creating a place for herself within the male-dominated world of 19th-century British academia.

Boole's most profound idea grew out of his interest in abstract algebra. In his book *An Investigation of the Laws of Thought on Which are Founded the Mathematical Theories of Logic and Probabilities,* he pioneered the concept of using algebraic equations to represent the methods of logical reasoning. He was the first to imagine variables taking "true" or "false" as values. These are now called "Boolean variables" in his honor, and they are a fundamental idea in computer programming. It is hard now to appreciate how shocking this was! It changed the way we think about logic, and even shaped the way we construct our sentences when we present a proof. Boole's colleague Augustus De Morgan, also an important logician, wrote,

> That the symbolic processes of algebra, invented as tools of numerical calculation, should be competent to express every act of thought, and to furnish the grammar and dictionary of an all-containing system of logic, would not have been believed until it was proved.

Boole got pneumonia in 1864 after getting caught in the rain, and his wife, who believed in homeopathic medicine, tried to treat it by wrapping him in wet blankets and pouring buckets of water over him. Sadly, that didn't work, and he died at the age of 49.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Boole/`

Nicolas Bourbaki

In 1934, a young André Weil was teaching calculus in Paris, and he thought the textbook was outdated. He and his friend Henri Cartan agreed to recruit a group of young mathematicians with modern attitudes and write their own book. Adopting the pen name of Nicolas Bourbaki,* the group started to list the topics they'd want to cover, and the project began to expand.

Early on, the committee of nine made a critical decision: their book would be complete and self-contained, including all of the necessary background material in order. They planned a series of six books, to be published a chapter at a time: Set Theory, Algebra, Topology, Functions of One Real Variable, Topological Vector Spaces, and Integration. Their ambitions would only expand: Armand Borel later wrote, "Already in September 1940, Dieudonné had outlined a grandiose plan in 27 books, encompassing most of mathematics."

Bourbaki's work was interrupted by World War II, when the invasion of France dispersed the group's members and took their attention away from mathematics. Participants retired from the project and younger members replaced them. From the beginning, every member had to agree on every decision! Borel described the process:

> [A draft] was read aloud line by line by a member, and anyone could at any time interrupt, comment, ask questions, or criticize. More often than not, this "discussion" turned into a chaotic shouting match.

Bourbaki's publications introduced new notation and terminology that is now standard, such as the symbol $\varnothing$ for the empty set and the terms "injective" and "surjective." Their axiomatic philosophy of mathematics was probably even more influential, and their unifying spirit, bringing out the connections among all areas of math, represents a vision that dates back to Euclid. Their writing style, strictly following the definition-theorem-proof format, and their bias towards proving results in the most general context possible, makes their work tremendously useful as a reference, but a difficult source from which to learn a new topic.

The group's membership has always been treated as an open secret, with participants steadfastly refusing to admit that Nicolas Bourbaki is not the name of an actual human.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Bourbaki/`

*The name is an inside joke. General Charles-Denis Bourbaki had led the Army of the East for the French Third Republic. In 1871, defeated by the Germans, Bourbaki tried to commit suicide to avoid surrendering, but his forehead was tougher than his bullet, which flattened against his skull! Much later, a student giving a parody lecture borrowed Bourbaki's name and those of other old French generals for the imaginary theorems he presented; Weil was in the audience.

Brahmagupta was a mathematician and astronomer, early in the golden age of mathematics in India. In 628, he revised and expanded the traditional manuals from which he'd studied, and published the *Brahmasphutasiddhanta* ("Correctly Established Doctrine of Brahma"). As is typical for the time and place, this text is entirely written in verse, using no mathematical notation, and it does not include any proofs. It does include a lot of new information, including one tremendously influential innovation: the first description of arithmetic involving zero as a number.

Brahmagupta

For Brahmagupta, zero was an extension of the natural number system, obtained by subtracting a number from itself. He presented rules for addition, subtraction, multiplication, and division involving zero in the *Brahmasphutasiddhanta*. Of course, this involved him in questions whose answers are not natural numbers. He handled subtraction from zero in terms of "fortunes" and "debts" — for example, "A fortune subtracted from zero is a debt." As for division by zero, he declared the value of $n \div 0$ to be $\frac{n}{0}$, with $\frac{0}{0} = 0$. [He didn't use this modern fraction notation.]

The quadratic formula appears for the first time in the *Brahmasphutasiddhanta*. Brahmagupta also solved some integer equations of the form $ax^2+c = y^2$, finding, for example, that $61x^2 + 1 = y^2$ for $x = 226153980$ and $y = 1766319049$, and he knew the formulas for sums of squares and cubes, $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ and $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.

In geometry, Brahmagupta found that a cyclic quadrilateral with side lengths a, b, c, and d and semiperimeter $S = (a+b+c+d)/2$ has area $\sqrt{(s-a)(s-b)(s-c)(s-d)}$, a formula which generalizes Heron's formula for the area of a triangle (which is the $d = 0$ case). He also proved a theorem that is now named after him: If $ABCD$ is a cyclic quadrilateral with perpendicular diagonals intersecting at X, then the line through X perpendicular to $\overline{AB}$ passes through the midpoint of $\overline{CD}$.

The *Brahmasphutasiddhanta* and Brahmagupta's other work, the *Khandakhadyaka* ("Edible Morsel"), were translated into Arabic in Baghdad in the 700s, introducing more people to the decimal number system, and popularizing what we now call Arabic numerals. We can assume that this development was a catalyst for the work of al-Khwarizmi, who lived in Baghdad, and who wrote *The Compendious Book on Calculation by Completion and Balancing,* the first genuine algebra text, around the year 825.

Image: Andreas Strick, `https://mathshistory.st-andrews.ac.uk/Biographies/Brahmagupta/`

Georg Cantor

Born in St. Petersburg, Russia, Georg Cantor moved to Germany with his family when he was eleven years old. He went to Zürich for college, where his father wanted him to study engineering, but eventually he got permission to pursue math instead, and transferred to the University of Berlin, where he learned from the important mathematician Karl Weierstrass, and from Leopold Kronecker, who would become his nemesis.

In his early career, Cantor solved a major problem on trigonometric series, which had defeated several famous people. To do this, he had to become deeply familiar with the properties of sequences and subsets in the set of real numbers. In the process, he developed a revolutionary new point of view on set theory, defining two sets to have the same size ("cardinality") when there is a one-to-one correspondence between their elements. This sounds reasonable, but has startling consequences: for example, he proved that $\mathbb{Z}$ and $\mathbb{Q}$ (the integers and the rational numbers) are the same size, though $\mathbb{Z}$ is a subset of $\mathbb{Q}$. The Cantor set, a fractal subset of the interval $[0, 1]$ which is the same size as $[0, 1]$ but does not contain any interval, was even more disturbing.

That is, Cantor built an entire spectrum of different infinite numbers, which was shocking. His "diagonal argument" showed that every set has more subsets than elements, and that $\mathbb{R}$ is larger than $\mathbb{Z}$. He conjectured that no set is larger than $\mathbb{Z}$ but smaller than $\mathbb{R}$: this is the *continuum hypothesis*, and its proof or disproof is still an open problem.

Until this point, the mathematical consensus had rejected the notion of quantities that were actually infinite. Philosophers as far back as Zeno of Elea, in the fifth century BCE, had seen the danger of paradox in infinite numbers, and any true paradox threatens the entire discipline of mathematics. Having built new foundations for the concept of infinity, Cantor had to overcome the weight of history to encourage others to adopt them. At the time, his most formidable opponent was his former teacher Kronecker, who took the extreme position that integers were the only numbers that truly existed. Cantor was forced to defend his work publicly; he also wrote several times to Pope Leo XIII and other priests, hoping to help the Church avoid mistaken doctrines regarding the infinite.

Throughout his life, Cantor struggled with depression, which naturally interfered with his research and his professional relationships. He died in a mental hospital during World War I — unable to work, poorly fed, and wishing he were at home. But David Hilbert, who was the world's most influential mathematician in the early 1900s, chose Biblical language to express his admiration of Cantor's work: "No one shall expel us from the paradise which Cantor has created for us."

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Cantor/`

Augustin-Louis Cauchy was born in Paris in 1789, the year when the French Revolution overthrew King Louis XVI and established the First Republic in France. His family knew both Lagrange and Laplace; this must have sparked Cauchy's interest in mathematics, and it gave him an exceptional opportunity to learn.

Augustin-Louis Cauchy

In mathematics, Cauchy's countryman Joseph Fourier had just set off a revolution of his own. Fourier's research into the flow of heat had led him to discontinuous functions that could be represented as infinite sums of continuous functions. This had never been imagined before, and it required new ideas. For the first time, mathematicians had to be perfectly specific about what they meant by words like "function" and "continuous."

In this new world, the ideas in Cauchy's masterpiece, the *Cours d'Analyse*, came into prominence. He introduces the idea of limits, as well as infinitesimals, in terms of variable quantities, then defines a function f to be continuous if, whenever α is infinitesimal, so is $f(x+\alpha) - f(x)$. (It was Weierstrass, in Germany, who rewrote this idea twenty years later in terms of real-valued variables, without infinitesimals, which is the version of continuity that we use today.) Using his new limit concept, Cauchy was able to give the modern definition of the derivative, as a limit of a difference quotient, and to define the definite integral as a limit of sums.

Cauchy also made huge strides in the world of complex analysis. He used the "Cauchy-Riemann equations," which were studied earlier by d'Alembert and Euler, to describe differentiable functions on complex domains, and applied the "Cauchy integral formula" to show that every function which is differentiable at and near a point in the complex plane can be represented as a power series — quite unlike the situation for real functions!

Cauchy's France was a place of intense political battles, and Cauchy was a strident partisan, mainly in support of the Catholic church, which had been taken over by the French government in the Revolution. He also treated other mathematicians disrespectfully, he was not a good lecturer, and he had trouble obtaining jobs — partly because he refused to swear an oath of allegiance to the government of the July Monarchy, which overthrew the pro-Catholic King Charles X, and partly because of his extremely poor personal relationships with others in the scientific community. Niels Henrik Abel, another of the great mathematicians of the 1800's, summed up the situation by writing,

> Cauchy is mad and there is nothing that can be done about him, although, right now, he is the only one who knows how mathematics should be done.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Cauchy/`

The mathematician who was born Francesco Cavalieri in Milan in about 1598 took Bonaventura as his name in religion when he became a Jesuat [sic] monk at the age of 17. He was soon transferred to a monastery in Pisa, where he learned mathematics from a teacher who had studied with the famous Galileo Galilei. In fact, Cavalieri was introduced to Galileo, and most of what we know about Cavalieri's life comes from the letters that he wrote to Galileo over the next two decades. It was Galileo's support that eventually won Cavalieri a job as a professor of mathematics at the University of Bologna, which he held for the rest of his life.

Bonaventura Cavalieri

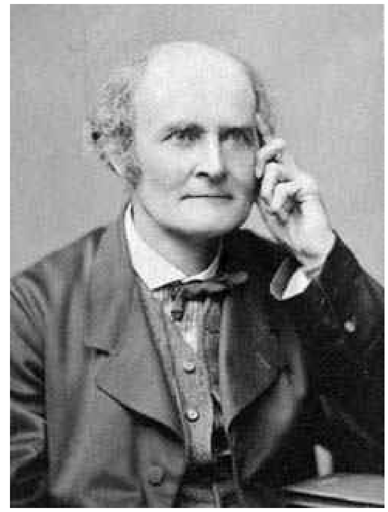
While he was trying to win an academic job, Cavalieri wrote the important book *Geometria indivisibilibus continuorum nova quadam ratione promota* (*The geometry of continuous indivisibles advanced by a new method*). It wasn't easy: the nineteenth-century French mathematician "Maximilian Marie suggested that if a prize existed for the most unreadable book, it should be awarded to Cavalieri for *Geometria*."* The method of indivisibles combined Archimedes's "method of exhaustion" with Kepler's work on infinitesimals in geometry. The basic idea is that any finite region in the plane can be subdivided into a collection of parallel segments, and that sliding these along lengthwise, independently of one another, doesn't change the area of the region. This, as well as the corresponding claim for solids, is now called "Cavalieri's principle," and this is what Cavalieri is best known for today.

It's hard to realize now, but in the 1600s it was profoundly controversial for Cavalieri to claim that a region with a finite positive area could be made out of line segments which have area zero. In 1632, the Jesuit Order, which was an important force in both religion and education, declared it unacceptable to believe or teach that a line was composed of points. It took until the late 1600s for Newton and Leibniz to publish the integral calculus which would ultimately resolve the debate, and Cavalieri's work was an important step on the way.

In 1639, Cavalieri published the book *Centuria di varii problemi* (*Hundreds of various problems*). In its full 22-word title, Cavalieri promises "to show the use and ease of logarithms in gnomics, astronomy, geography, altimetry, planimetry, stereometry, and practical arithmetic." The subject of "planimetry" refers to measurements in the plane, such as angles, distances, and areas, and it is here that the approximate integration formula that we now call "Simpson's rule" was published for the first time, seventy years before the birth of Thomas Simpson. He also published two tables of logarithms, which were influential in promoting the use of logarithms in Italy.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Cavalieri/`

*Kristi Andersen, "Cavalieri's Method of Indivisibles," Archive for History of Exact Sciences 31(4): 294.



Arthur Cayley

Arthur Cayley's grandfather was the British Consul General to Russia under Empress Catherine the Great, and Arthur lived in St. Petersburg until he was seven years old. Moving to London in 1828, he was a very advanced student, and was accepted to study at Cambridge.

While at Cambridge, Cayley published three papers. The very first of these, in 1841, is called "On a theorem in the geometry of position." Cayley immediately explains the formula for the determinant of $n \times n$ matrix (though these words were not yet in use), states that for 3×3 matrices $\det(AB) = (\det A)(\det B)$, and that this theorem can be generalized, and uses determinants to give tests for whether four given points lie on a circle, and whether five given points lie on a sphere, based only on the straight-line distances among them.

Graduating from Cambridge with a first-class degree in mathematics made Cayley a "Wrangler," and in fact he was Senior Wrangler: he finished with the highest grades in his graduating class. He won a Fellowship, which entitled him to teach at Cambridge for four years. (To become a tenured professor, he would have had to take vows as a priest.) He used this time to publish more work on many subjects, including determinants, and started to work with George Boole by correspondence.

When his Fellowship ended, Cayley took a job as a lawyer, which he hated, and he spent his free time discussing mathematical research with J. J. Sylvester and others. During that time, Cayley published very important work on matrix theory, showing for the first time how to multiply matrices and defining the inverse matrix. In other papers, he defines the concept of an abstract group for the first time, presenting examples of "Cayley tables" and bringing together the ideas of matrices, quaternions, and permutations. These were considered radically different subjects until Cayley's Theorem, which means that you can use a clever encoding to think of every abstract group as a group of permutations.

Cayley finally got a job as a professor at Cambridge in 1863. He was finally able to concentrate on mathematics full time, and ended up publishing 900 papers. However, his students didn't think he was a good teacher. At the time, mathematical education at Cambridge focused heavily on the Tripos, a series of competitive exams that students had to pass to graduate, and Cayley often lectured on subjects that wouldn't be on the test. He valued mathematics for its own sake, saying:

> As for everything else, so for a mathematical theory: beauty can be perceived but not explained.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Cayley/`

Gabrielle-Émilie Le Tonnelier de Bretuil was born into an upper-class family: her father was Principal Secretary and Introducer of Ambassadors in the royal court of King Louis XIV. Tutors taught her languages, classical literature, philosophy, and mathematics, as well as riding, dance, and fencing.

Émilie du Châtelet

At nineteen, she entered an arranged marriage with a military officer, politician, and nobleman whose title was the Marquis of Chastellet. It was her marriage, and the flexibility of spelling in pre-revolutionary France, that changed her name to Émilie, Marquise du Châtelet. In the end, they agreed to live independent lives, so she was able to spend her time gambling at cards, conducting romances, and studying mathematics and philosophy, with Clairaut among her teachers. As a woman, she was excluded from the Paris Académie where science was discussed, but individuals were willing to work with her. She wrote:

> I feel the full weight of the prejudice which so universally excludes us from the sciences.... Chance made me acquainted with men of letters who extended the hand of friendship to me. ...I then began to believe that I was a being with a mind.

In 1733, soon after the birth of her third child, du Châtelet met the great writer Voltaire, who was ten years older than she was. They were constant social companions, and before long, she invited him to live at her country house — he needed to lie low for a bit after publishing some essays that were too favorable to the British system of government, and too hostile to the religious establishment. Together, Voltaire and du Châtelet wrote a book that introduced Newton's scientific discoveries to France, and she followed this with her own *Institutions de Physique*, in which she brought together Newton's methods with Leibniz's argument that the *"vis viva"* mv^2 was an important quantity conserved in many mechanical systems, rather than the momentum mv emphasized by Newton. This was an important step toward discovering the concept of kinetic energy.

After this, du Châtelet began her most influential work: the first French edition of Newton's *Principia*, in which he introduced calculus. This was not just a translation: she added commentaries that explained the theory and made it accessible. Very few people then could have understood this material and applied it, as she did, to the astronomy of the solar system, including Kepler's laws of planetary motion. Though not published until after her death, it freed French mathematicians from the outdated theories of Descartes.

Du Châtelet died at the age of 42 from complications of childbirth.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Chatelet/`

Alexis Clairaut

Alexis Clairaut was born in Paris in 1713. Tragically, though he was one of twenty siblings, he was the only one who didn't die in childhood. His father, a math teacher, drove him incredibly hard: he learned to read from Euclid's *Elements* and was studying calculus, from L'Hôpital's textbooks, by the time he was ten years old. He presented his first original paper at the age of 13.

His name is familiar to calculus students thanks to Clairaut's theorem: If the second partial derivatives of z are continuous near a point, then changing the order of differentiation doesn't change the value of the mixed second partials, and $z_{xy} = z_{yx}$ at that point. Clairaut published this theorem in 1740 in the *Memoirs of the Paris Academy of Sciences*, but it was still too early for a proof that we'd consider valid now. The notion of continuity wouldn't be fully developed for another 75 years, and Clairaut probably didn't realize that it had to be assumed, or that it might not be true in every example. Moreover, Clairaut himself wrote, in that 1740 paper,

> I am not the only one who has found this theorem. M. Fontaine had also found it... and M. Euler, the famous mathematician, has given the Academy of Petersburg, in the volume which is now in press, a piece full of fine work on the integral calculus, where he uses this same discovery.

Hermann Schwarz gave the first totally successful proof of Clairaut's theorem in 1873.

The author Voltaire and his partner, the Marquise Émilie du Châtelet, were good friends of Clairaut's, and Clairaut helped with her mathematical training. It was du Châtelet who first translated Newton's great *Principia Mathematica* into French, and she included a number of Clairaut's discoveries along the way.

Clairaut was most interested in applying math to physics and astronomy, so much so that he spent a year on an expedition to Finland with several other scientists, including Anders Celsius. They set out to measure the length of one degree of longitude, far north of their home, and verify Newton's conclusion that the Earth is approximately an oblate spheroid (wider than a sphere near the equator and narrower toward the poles). Later, he studied the orbit of the moon. For a while, he thought he had disproved Newton's inverse-square law of gravity, but soon realized that errors due to approximation were keeping the theory's predictions from matching his observations. He also studied optics, and he published very successful textbooks on geometry and algebra.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Clairaut/`

When asked what he wanted to be when he grew up, eleven-year-old John Conway said that he wanted to be a mathematician at Cambridge. Indeed, he earned his bachelor's degree and doctorate at Cambridge, and was named a fellow there in 1964. He worked on number theory, logic, and algebra, but also loved games — especially backgammon.

John Horton Conway

Conway made a reputation by discovering three new simple groups: a significant step toward the full classification. At almost the same time, he built on an idea of John von Neumann to create the Game of Life. Its small set of simple rules, perfect for the new age of personal computers, made patterns on a square grid evolve into surprising complexity.

Throughout his life, Conway studied a diverse array of problems, bringing to all of them not only deep mathematical insight but also a talent for clear, interesting writing and a gift for clever names. His interest in games led to his invention of the "surreal numbers," a class including not only the real numbers, but also infinitely large numbers and infinitely small ones: in fact, all of the numbers that could possibly belong to an ordered field. Astonishingly, each surreal number is defined to be a position in some two-player game!

Conway enjoyed studying recreational math, not restricting himself to "serious" problems. He fully analyzed "audioactive" puzzle sequences like $(2, 12, 1112, 3112, 132112, \ldots)$, where the challenge is to predict the next term. After a few preliminaries, Conway proves his "cosmological theorem:" The terms in such a sequence eventually split into non-interacting subsequences ("atoms"), for which there are exactly 92 possibilities, and Conway names these for the 92 chemical elements from hydrogen to uranium. The number of digits in the terms grows exponentially: In the limit, each term is λ times as long as the term before, where $\lambda \approx 1.303577\ldots$ is "Conway's constant."

Conway's coauthor Richard Guy drew this memorable portrait:

> Conway is incredibly untidy. The tables in his room... are heaped high with papers, books, unanswered letters, notes, models, charts, tables, diagrams, dead cups of coffee and an amazing assortment of bric-à-brac, which has overflowed most of the floor and all of the chairs, so that it is hard to take more than a pace or two into the room and impossible to sit down.... In spite of his excellent memory he often fails to find the piece of paper with the important result that he discovered some days before, and which is recorded nowhere else.*

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Conway/`

*Albers and Alexanderson (eds.), *Mathematical People: Profiles and Interviews* (1985), 43–50.

Richard Dedekind

Julius Wilhelm Richard Dedekind, son of a law professor, grew up in Brunswick, Germany, and got a good education, choosing to focus on math because of its logical structure. At the University of Göttingen, he had Gauss for a professor and Riemann for a classmate. Though he earned his doctorate and completed the habilitation degree that made it possible for him to get a job as a professor in Germany, he did not feel confident in his preparation level, and continued to take courses even after being hired to teach at Göttingen.

Dedekind was soon joined by Lejeune Dirichlet, who was hired when Gauss died. A generation older than Dedekind, he helped fill the gaps in the younger professor's knowledge. With a letter of recommendation from Dirichlet as a good teacher, Dedekind got a job in Zürich. There he was assigned to teach calculus for the first time in 1858, and while planning his lectures, he saw a new way to define what a real number is. Assuming that the rational numbers are understood, Dedekind declared that a real number can be represented as a non-empty proper subset x of the rationals with no greatest element, with the property that every rational number less than any element of x is also an element of x.

Such a subset is now called a Dedekind cut; this construction is one of the first successful efforts to say clearly what a real number is. The motivation came from the crisis in calculus, which had created a demand for rigorous definitions of continuity and derivatives, given in terms of arithmetic rather than geometry. Dedekind did not publish his work on cuts until 1872, but people were strongly interested in the question: Hamilton, Cantor, and others all published their own constructions of the reals at almost the same time.

Dedekind took up the important task of collecting and publishing the complete works of Dirichlet, Gauss, and Riemann after they died. In doing so, he studied Dirichlet's research on algebraic number theory, which stimulated him to write his own book on the subject. While carrying out this line of study, he defined the concepts of "fields" and "ideals" — both essential definitions in abstract algebra.

Pursuing the theme of justifying the existence of various types of numbers, Dedekind published *What are numbers and what are they for?* in 1888, where he showed how to start from the natural numbers to construct first the integers, and then the rational numbers. In every case, he considers these sets as concrete objects actually containing an infinite number of elements, which was a controversial idea! In addition, he gives his own justification for the reasoning involved in a proof by mathematical induction. Dedekind was also a role model for his writing style itself, described by Hermann Minkowski as "a minimum amount of blind calculation, a maximum of clear-seeing thought."

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Dedekind/`

Augustus De Morgan was born in India to a British soldier and his wife. As an infant, an infection cost him the sight in his right eye, and his classmates later bullied him due to his disability. He went to Cambridge, where besides mathematics, he considered careers in religion, medicine, and law. He was also an excellent flute player.

Augustus De Morgan

He was studying for the bar exam when the brand new London University appointed him to their chair of mathematics. Right away, he translated a French algebra textbook for his students — actually, only the first few chapters, because, he explained, "every one who is desirous of attaining a considerable degree of mathematical knowledge must become acquainted with the French language."

After working at London University for eight years, De Morgan resigned in a protest for academic freedom. The school rehired him five years later, and he remained there for thirty more years, but finally resigned again when his college decided against hiring a candidate based on their religious beliefs.

De Morgan is best known for his work on logic. He was the first to state De Morgan's laws in Boolean algebra — that the negation of "A and B" is "not A or not B," and vice versa. He coined the term "mathematical induction" in 1838, becoming the first to formalize that proof technique in a systematic way. He gave individual lessons in math and logic to Ada Lovelace, who is regarded as the first computer programmer (a century before electronic computers were invented). He wrote hundreds of encyclopedia-style articles introducing math and other topics to the general public, and he published textbooks on arithmetic, trigonometry, calculus, and the complex numbers.

In an 1852 letter to his friend William Rowan Hamilton, De Morgan proposed a new claim that quickly became famous, and attracted many false proofs across the decades:

> A student of mine [Frederick Guthrie, inspired by his brother Francis] asked me to day to give him a reason for a fact which I did not know was a fact — and do not yet. He says that if a figure be any how divided and the compartments differently coloured so that figures with any portion of common boundary line are differently coloured — four colours may be wanted, but not more.... Query cannot a necessity for five or more be invented....

The celebrated "four color problem" was only solved in 1976, by Kenneth Appel and Wolfgang Haken of the University of Illinois, with the help of a computer.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/De_Morgan/`

The philosopher René Descartes was a critical figure in the era now known as the Scientific Revolution. Rejecting the traditional reverence for the wisdom of ancient authorities (especially Aristotle), he set out to develop his own canon of reliable knowledge, starting from scratch. In his *Discourse on the Method*[*], he writes,

> The first [logical principle] was never to accept anything for true which I did not clearly know. . . and to comprise nothing more in my judgment than what was presented to my mind so clearly and distinctly as to exclude all ground of doubt.

René Descartes

When anyone attempts to organize all of their knowledge systematically, the key question is where to begin. How can the very first claim be justified, with no prior knowledge to reason from? After acknowledging that any or all of his beliefs might be mistaken, Descartes decided that he could be absolutely certain of at least one thing: he must exist, since otherwise he couldn't be thinking at all.

> I observed that this truth, 'I think, therefore I am,' was so certain and of such evidence that no ground of doubt, however extravagant, could be alleged by the sceptics. . . .

That's probably the most famous quotation in all of philosophy, but his work in mathematics turned out to be more important! He argued that it was natural to study math first, since "of all those who have hitherto sought truth in the sciences, the mathematicians alone have been able to find any demonstrations, that is, any certain and evident reasons. . . ." For the first time, Descartes introduces the idea of naming points in the plane using pairs of real numbers (though the details would have to wait for later authors). This coordinate system, called "Cartesian" in his honor (since his name was printed in Latin texts as Renatus Cartesius), formed a link, for the first time, between algebra and geometry! Take a moment to think about this: the classical Greeks understood both parabolas and square numbers in depth, but connecting those concepts by graphing the equation $y = x^2$ was impossible until Descartes published *La Géométrie* two thousand years later.

As you might expect, the new tools that Descartes provided led directly to a new era of discovery in mathematics and science. From the publication of *La Géométrie* in 1637, it is just under thirty years to Newton's invention of calculus in 1666.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Descartes/`

[*] In full, *Discourse on the Method of Rightly Conducting the Reason and Seeking Truth in the Sciences.*

Paul Dirac grew up in England, and was about to turn 12 at the outbreak of World War I. Thus he was too young for military service, but many of the older boys were gone, which left some extra educational opportunities open at his school. He studied electrical engineering at his hometown university; he had been admitted to Cambridge, but couldn't afford the tuition.

Paul Dirac

However, once he graduated, he did go on to study at Cambridge, and there he found his calling: creating the mathematical foundations for quantum theory. The key idea for Dirac's thesis came while he was studying a revolutionary paper by Werner Heisenberg. In quantum theory, observable phenomena are represented by functions, and Dirac saw that Heisenberg's uncertainty principle could be expressed by saying that the position and momentum functions did not commute.

In 1927, shortly after he earned his Ph.D., Dirac published a paper called "The Physical Interpretation of the Quantum Dynamics," in which he described a quantum operator as a kind of matrix with uncountably many rows and columns. To work with such a matrix, he said, he had to invent the Dirac delta function:

> One cannot go far... without needing a notation for that function of a [real] number x that is equal to zero except when x is very small, and whose integral through a range that contains the point $x = 0$ is equal to unity. We shall use the symbol $\delta(x)$ to denote this function.... Strictly, of course, $\delta(x)$ is not a proper function of x.... All the same one can use $\delta(x)$ as though it were a proper function for practically all the purposes of quantum mechanics without getting incorrect results.

Dirac's work was highly honored from the very beginning. At the age of 30, only six years after earning his Ph.D., he became Lucasian professor of mathematics at Cambridge, a chair which was previously held by Newton, and would later belong to Stephen Hawking. The following year, Dirac shared the Nobel Prize for Physics with Erwin Schrödinger.

Dirac was not talkative, maybe because his Swiss father had only allowed the family to speak to him in French. The other professors at Cambridge "honored" Dirac by naming a new unit after him: one dirac is equal to one word per hour. Dirac was anti-religious, arguing that religion was promoted mainly to control dissatisfaction among the exploited poor. The physicist Enrico Fermi teased him about this: "Well, our friend Dirac has got a religion and its guiding principle is 'There is no God, and Paul Dirac is His prophet.' "

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Dirac/`

Lejeune Dirichlet

Lejeune ["Junior"] Dirichlet grew up in Düren, which is now in Germany but was then under the control of the new Emperor of France, Napoleon Bonaparte. He was an enthusiastic math student, taking classes from Ohm in Cologne before moving on Paris, where he worked with famous mathematicians like Laplace, Legendre, and Fourier. He eventually became a professor in Berlin, and married his wife Rebecca, whose brother was Felix Mendelssohn. His reputation grew to the point that when the great Gauss died, Dirichlet replaced him at the University of Göttingen.

As part of Dirichlet's work in number theory, he took an interest in the challenge of approximating an irrational number α by rational numbers with small denominators. By looking at the decimal forms of multiples of α, he found a clever way to prove that for any N, there is always a rational number $\frac{p}{q}$, with $q \leq N$, for which $\frac{p}{q}$ is within $\frac{1}{qN}$ units of α. Since then, his technique has been applied to an extremely broad range of problems, and is sometimes known as the Dirichlet principle, but we more often use the name that he himself gave it — the pigeonhole principle.

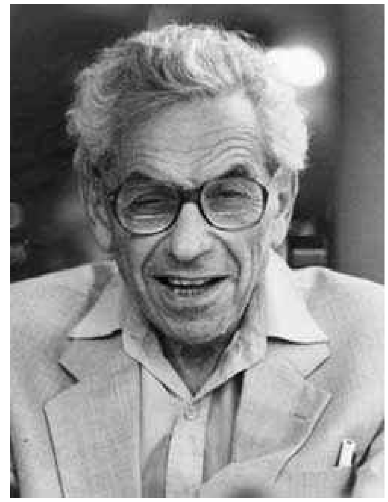
In fact, it was Dirichlet who, in 1837, first defined functions in the way that we now understand them: a rule assigning a unique output to every input. This was not clear to anyone at the time! Actually, it was controversial for decades: mathematicians were used to just thinking of functions as curves, or as formulas, which was a problem: from that viewpoint, it was hard to be specific about the meaning of some properties of functions, like continuity. Dirichlet was led to his definition through his work on the convergence of sums of functions: he solidified the theory of Fourier series by finding the right way to fill the logical gaps in prior work of Fourier, Poisson, and Cauchy.

Also in 1837, Dirichlet announced his most famous achievement, now known simply as Dirichlet's theorem: Unless a and b share a prime factor, the sequence of integers $a, a+b, a+2b, \ldots$ always contains infinitely many prime numbers. This was a major accomplishment! Legendre had conjectured this at least 35 years earlier, and Gauss knew about the question, but to find his proof, Dirichlet basically had to invent a new field of mathematics, now called analytic number theory.

The 1850 journal of a younger mathematician, Thomas Hirst, describes Dirichlet:

> He was unwashed, with his cup of coffee and cigar. One of his failings is forgetting time, he pulls his watch out, finds it past three, and runs out without even finishing the sentence.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Dirichlet/`



Paul Erdős

Paul Erdős (AIR-dish) was born in Hungary, to parents who were both math teachers. He started college at 17, despite the oppressive rules of Hungary's pro-Nazi government: Erdős came from a Jewish family. Four years later, he had earned a Ph.D., but still had to leave Hungary to find work. As World War II approached, he made his way to the U.S. However, he was blocked from returning during the Red Scare of the 1950s, when he refused to say that he would never go back to Hungary.

Ultimately, Erdős became a nomad. He traveled, often unannounced, from one mathematician's home to another, declaring, "My brain is open," and collaborating on research for a few days at a time, relying on his hosts for meals, laundry, and transportation. No one has ever worked productively on math with so many other people: he published over 1500 papers, with 511 different co-authors. These distinguished mathematicians are said to have Erdős number 1, their other co-authors have Erdős number 2, and so on.

While still a university student, Erdős found an elegant new proof of a well-known theorem on the distribution of prime numbers: "Chebyshev said it, and I say again — There is always a prime between n and $2n$." In 1949, with Atle Selberg, he found the first elementary proof of the prime number theorem: the number of primes between 1 and n is approximately $n/\ln n$. Throughout his life, he strove to find beautiful solutions to beautiful problems. It was Erdős who first imagined "The Book," in which God keeps the most elegant proof of every mathematical theorem (and, according to Erdős, sometimes deliberately conceals them).

He also posed significant problems that shaped the direction of research in combinatorics and number theory, frequently offering (and paying) cash bounties for solutions. Though he rarely held a regular job, Erdős was confident:

> What would the world's strongest bank do if all its depositors came in on the same day and demanded their money? Of course the bank would go broke. But that is much more likely than that all of my problems will be solved.*

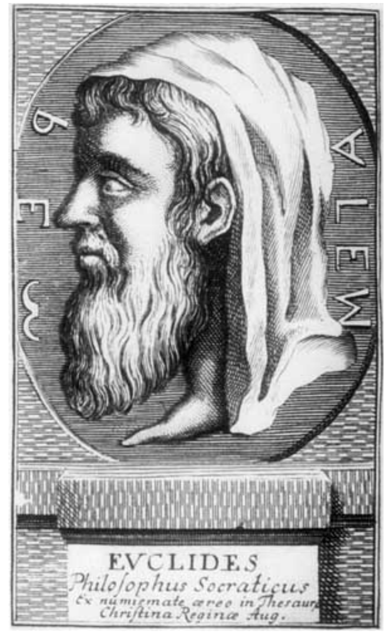
Despite his death, hundreds of these prizes are still active.

Erdős died in 1996, at a conference in Poland, and was buried in Hungary. He proposed that his gravestone should read, "I've finally stopped getting dumber."

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Erdos/`

*Steven G. Krantz, *Mathematical Apocrypha Redux*, 128.

Euclid — Greece/Egypt, c. 325–265 BCE



Euclid

Euclid of Alexandria belongs to the classical era of Greek culture, but he lived and taught in the great city of Alexandria, in Egypt. Since he lived over two thousand years ago, it isn't surprising that we don't know much about him for sure. In fact, don't take the picture at right too seriously: We can't even be completely sure that Euclid was an actual person! Though it seems most likely that Euclid was a mathematician in Alexandria around 300 BCE, we might instead imagine a group working together on a massive project in the field of geometry, and borrowing a name from the earlier philosopher Euclid of Megara (a student of Socrates). Certainly, these two Euclids have often been confused with each other over the years.

Euclid is known for writing the *Elements*, the single textbook on which European math education was based from the time it was written until well into the twentieth century. In the *Elements*, Euclid arranged all of the mathematics that was known in his time and place into a single coherent account, and showed how to derive all of it from a short list of assumptions. Euclid wasn't the first to discover the mathematics that he wrote about, but his work became the model of how math is done, proceeding from axioms and definitions to propositions and theorems via rigorous proof. Though later mathematicians found some places where Euclid's proofs used unstated assumptions, his work stood up astonishingly well to close scrutiny. For two thousand years mathematicians and philosophers treated the *Elements* as a accurate description of the physical nature of the universe — or even more: a logical exposition of what the universe had to be like, and why.

The proofs in the *Elements* are based on five postulates — the initial assumptions that must be made in any system of deductive reasoning. [You can never prove everything, because when you try to prove the first fact, you don't know any facts to start from!] Of the five, one (the "parallel postulate") has always been considered less obvious than the others, and ever since Proclus (400s CE), there have been attempts to prove it using the other four. But in the 1800s, mathematicians like Gauss, Poincaré, Lobachevsky, Bolyai, and Riemann showed that it's reasonable, and useful, to interpret geometry in such a way that the parallel postulate is actually false! This "non-Euclidean geometry" led directly to a revolution in the way people thought about truth and reason. Rather than thinking of postulates as self-evident true statements about reality, we now regard them as hypotheses, which may be applicable in some contexts but not in others, and which set limits on the meaning of the language that we use in our reasoning. Euclid's dream of arranging all mathematical truth in a single unified presentation now seems to be doomed, but his vision continues to shape our search for knowledge.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Euclid/`

Leonhard Euler

At the age of 14, Leonhard Euler became a student of the famous mathematician Johann Bernoulli at his hometown school, the University of Basel. He earned his master's degree there, officially in philosophy. His father wanted him to do a doctorate in theology, but he got permission to switch to mathematics, earned his degree, and in 1727 he got a job teaching in Saint Petersburg, Russia, at a new scientific institute founded by Catherine the Great. He moved to Berlin in 1741 to work for King Frederick the Great, who was then founding the Berlin Academy of Science, then returned to Saint Petersburg in 1766. Soon after this, he became ill, and lost his eyesight; he had lost all vision in his left eye by about 1740, and was completely blind by 1771. Still, he continued his mathematical research until the day he died, 12 years later, and remained so productive that his work continued to be published until 1830, when he had been dead for 47 years.

Indeed, Euler was the most productive mathematician of all time, publishing 785 catalogued papers, and one of the most influential. Much of the mathematical notation that we take for granted was invented by Euler: examples include e, i, $\sin x$, $\sum_{k=1}^{n}$, and $f(x)$. He worked in all areas of mathematics; it is likely that he was among the last humans to know all (or essentially all) of the mathematics that was known at any given time. (Gauss is probably the only later mathematician who might plausibly have claimed this achievement.)

Euler proved, or defined, Euler's formula for complex exponents: $e^{ix} = \cos x + i \sin x$. He invented the field of graph theory, and proved another theorem that is now named after him: If a polyhedron has V vertices, E edges, and F faces, then $V - E + F = 2$. He solved what we now call the Basel problem by showing that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, and went beyond the knowledge that there are infinitely many prime numbers to prove that $\sum \frac{1}{p} = \infty$, the sum running over all positive primes p. Besides $e = 2.7182818\ldots$, which is named in his honor, there is another real number called Euler's constant:

$$\gamma = 0.57721566\cdots = \lim_{n\to\infty} \left[\left(\sum_{k=1}^{n} \frac{1}{k} \right) - \ln n \right].$$

There's no doubt that Euler would belong on the "Mount Rushmore of mathematics," as one of the four greatest mathematicians of all time. Who else would you choose?

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Euler/`

Pierre de Fermat earned his living as a lawyer, but had a passion for mathematics. He shared this interest with a coworker, who took news of the things Fermat had done to the mathematicians in Paris. From then on, Fermat corresponded regularly with other French mathematicians, including Marin Mersenne, generally to challenge them to solve problems he'd already worked out. When Mersenne couldn't solve most of these problems, he asked Fermat how to do them. Fermat responded with two essays — one on the topics that would soon belong to differential calculus, and another on applying algebra to problems in geometry. The first of these essays contains the original statement of ”Fermat's theorem” that the local maxima and minima of a function could only occur at critical points.

Pierre de Fermat

Regrettably, Fermat fell into a feud with René Descartes after expressing a negative opinion of one of Descartes's essays on the refraction of light. After other mathematicians got involved, Descartes acknowledged the quality of Fermat's work (while criticizing his exposition of it), but he continued to denigrate Fermat's ability privately, in a way that seriously damaged Fermat's reputation. Years later, reviewing the controversy, Fermat proposed that light travelling from one point to another always follows the shortest possible path between them, used this hypothesis to re-prove Snell's law on refraction, and suggested treating it as a fundamental axiom of optics.

Later in his life, in a short exchange of letters, Fermat reviewed the work of a younger French mathematician, Blaise Pascal, and together they established the basic framework for a mathematical theory of probability.

Five years after Fermat's death, his son published an edition of *Arithmetica* by Diophantus annotated with the comments that Fermat had written in his own copy of the book. One of these taunted the mathematicians of the world for centuries: Fermat wrote,

> To divide a cube into two cubes, a fourth power into two fourth powers, or in general any power above the second into two powers of the same name is impossible, and I have certainly found a remarkable proof of this. But that won't fit in this small margin.

The fact that no one could find such a proof of "Fermat's last theorem," until Andrew Wiles did it in 1994 by methods that could not have been imagined even in 1900, suggests that Fermat was mistaken. However, the search for his lost proof was one of the driving forces that inspired mathematical progress for over three hundred years.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Fermat/`

Leonardo Pisano (Fibonacci)

Leonard of Pisa grew up in what is now Algeria. His father had moved there for professional reasons: he represented traders from the cathedral city of Pisa (where a new bell tower was under construction, and the soil was just beginning to sink under the foundations).

At the time, science was barely progressing in Europe: "About the only mathematics that was carried out was that necessary for the computation of the date of Easter."* However, the Islamic world, including Algeria, was in a golden age. Through his immersion in this scholarly culture, Fibonacci learned state-of-the-art mathematical methods, especially the place-value system based on the Arabic numerals $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$.

When he got back to Italy, Fibonacci published his *Liber Abaci* ("Book of the abacus,") which changed the world by bringing the Arabic numerals to European scientists: a dramatic increase in efficiency compared to the Roman numerals they had been using. The text includes many practice problems, including the one for which Fibonacci is best remembered:

> A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?

If no rabbits ever die, then the number of pairs after n months is $a_n = a_{n-1} + a_{n-2}$ (though Fibonacci did not use this notation, since variables and the equals sign would not be invented for centuries): the total number of pairs from the previous month, plus one new pair for each pair that's at least two months old. The problem suggests $a_0 = 1$ and $a_1 = 1$, from which the sequence of "Fibonacci numbers" $1, 2, 3, 5, 8, 13, 21, \ldots$ follows.

In 1225, Fibonacci wrote *Liber quadratorum* ("Book of squares"), more advanced than *Liber Abaci* though less influential, in which he proved several theorems in number theory relating to the square numbers $1, 4, 9, 16, 25, \ldots$. For example, he showed that any square is the sum of the first several odd numbers $1 + 3 + 5 + \cdots + (2n - 1) = n^2$, found the sum of the squares of consecutive numbers $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ and of the squares of consecutive odd numbers $1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2 = \frac{n(2n-1)(2n+1)}{3}$, and proved that $x^4 - y^4$ cannot be a square.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Fibonacci/`

*Victor J. Katz, "The mathematical cultures of medieval Europe," *History and Pedagogy of Mathematics* (July 2016).

Joseph Fourier

The twelfth of fifteen siblings, Joseph Fourier was orphaned at the age of ten. He considered the priesthood, but opted for a secular life in mathematics. Politically, he became a partisan of the French Revolution, though he tried to withdraw from his involvement when the Reign of Terror got underway. After taking sides in a post-revolution political struggle, he was arrested, and would likely have been guillotined if Robespierre hadn't beaten him to it.

Fourier took classes from Lagrange and Laplace (he liked Lagrange better), and within a few years, he took over Lagrange's chair at the École Polytechnique in Paris. He soon left, though, when he was appointed scientific advisor to Napoleon's invasion of Egypt. The French army was successful, but their navy was not, and Fourier was stranded in Egypt until the French surrender in 1801. While there, he helped to found the Cairo Institute, a scientific society.

When he got back to France, Fourier wanted to return to his university, but Napoleon appointed him Prefect of the Department of Isère, the local official in charge of state services. This took him to the city of Grenoble, where he did his most important research. Between 1804 and 1807, while working for the government, he wrote a paper called *On the Propagation of Heat in Solid Bodies,* in which he introduced "Fourier's law of heat conduction," and used it to deduce the "heat equation":

$$\frac{\partial u}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

This differential equation models how heat diffuses through a solid over time.

This was also the paper in which he introduced Fourier series. He concluded that every function can be written as an infinite sum of sines and cosines. [It turns out that this actually isn't true for *every* function, but the necessary conditions would not be understood for a long time.] This was so unexpected that it became controversial: his former teachers Laplace and Lagrange concluded that it was simply wrong. In particular, Fourier was interested in discontinuous functions, and no one expected that a sum of continuous functions could be discontinuous.

The resulting debates had a profound impact on the development of real analysis, as everyone worked to salvage what they could of their earlier intuitions, clarifying the right definitions for ideas like convergence and continuity. Beyond that, Fourier series also became an incredibly powerful tool in an immense range of practical applications, from audio noise reduction to MRI imaging to data compression (including MP3 audio, JPEG images, and cellular network transmissions).

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Fourier/`

When you were eight years old, how long would it have taken you to add up all of the whole numbers from 1 through 100? The young Carl Friedrich Gauss astonished his teacher by finding the sum instantly:

$$\begin{aligned}1 + \cdots + 100 &= (1 + 100) + (2 + 99) + \cdots + (50 + 51)\\ &= 50 \cdot 101 = 5050.\end{aligned}$$

His father would have been less surprised: 3-year-old Carl had spotted an error in his dad's calculations!

Johann Carl Friedrich Gauss

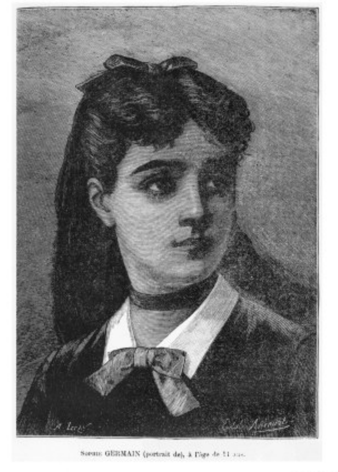
Gauss was able to focus on math starting when he was 14, thanks to financial support from the local duke. Before he was 18, he developed the principle of least squares optimization, and he correctly predicted how common prime numbers of any given size are (though the "prime number theorem" would not be proved until 40 years after Gauss's death). At the University of Göttingen, he made two tremendous advances that established him as already a first-class mathematician. First, he listed all n for which the regular n-gon can be constructed with a compass and straightedge (his list was proved to be complete by Wantzel later) and showed how to construct the regular 17-gon, and he gave the most complete proof of the fundamental theorem of algebra that was then known (Argand came up with the first fully rigorous proof within a decade).

In 1801, a monk in Sicily saw something that looked like a comet, but "might be something better," and named it Ceres. [It's either the first asteroid or the first dwarf planet ever found.] He published his observations, but by the time they came out, Ceres had gone behind the sun, and astronomers needed a prediction of its location to find it once it came back out. Gauss was the only one who could tell them where to find it. Several years later, he won a prestigious job as director of the observatory in Göttingen. As an astronomer, he had to deal with measurement errors, and to do so he used the normal (or "Gaussian") distribution as a model. While working out the gravitational force exerted by a ellipsoid, he uncovered some special cases of a property of surface integrals; the full divergence theorem ("Gauss's theorem") was proved later by Ostrogradsky.

Gauss was an unhappy man. He had poor relationships with his father and with his sons. His patron died in battle, fighting Napoleon, and his wife died after having three children in three years. He never collaborated with other mathematicians, partly because he was a German nationalist while the greatest mathematicians of the time were in France, and partly because he simply didn't want to give any clues to others who might manage to prove things before he did. When people wrote to him about mathematics, he was politely hostile, usually telling them that he already knew everything they had proved, but hadn't bothered to publish it. His behavior made him less influential than he should have been, but he still ranks among the most important mathematicians of all time.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Gauss/`

Sophie Germain was born in Paris, the second of three girls. Her parents educated her at home, but this was interrupted when she was thirteen by the French Revolution — her father held political office. Through her teen years, Germain had to educate herself, because her parents forbade her to learn math: they thought it was inappropriate for girls. She studied when she was supposed to be sleeping, against the will of her parents, who tried to prevent her by taking away the lights from her bedroom, then the heat, and finally even her clothes, so she couldn't sneak out and do math. When they found her asleep at her desk, wrapped in blankets, in a library so cold that her ink was frozen, they realized that they had lost the battle.



Sophie Germain

When the École Polytechnique, now a highly prestigious university, opened in Paris in 1794, Germain was eighteen years old. They didn't admit women, but some of the students shared their lecture notes, and so Germain took analysis from Lagrange without his knowing it. In fact, she submitted a paper at the end of the term, borrowing the name of a student (M. LeBlanc) who had dropped out. The paper was original enough that Lagrange sought her out, and became an advisor to her (though she was still not admitted to the school).

Germain also exchanged letters with Gauss, again using her pseudonym. In 1806, with France and Germany at war, the French had invaded Gauss's town, and she tried to protect Gauss by reaching out to a French general who was a family friend. An officer was sent to find Gauss, who was indeed safe, but didn't recognize the name of the woman who had intervened on his behalf. The truth came out, and Gauss (who was not generous with praise) described her as

> ... a brilliant example of what I would find difficult to believe.... [W]hen a woman, because of her sex, our customs and prejudices, encounters infinitely more obstacles than men... yet overcomes these fetters and penetrates that which is most hidden, she doubtless has the noblest courage, extraordinary talent, and superior genius.

Germain won a prize from the Paris Academy of Sciences, who had issued a challenge to develop a theory explaining how surfaces vibrate, by deriving a fourth-order partial differential equation. But she spent most of her life trying to prove "Fermat's last theorem" — if $n > 2$ is an integer, then there are no solutions in positive integers to the equation $x^n + y^n = z^n$, and succeeded when $3 \leq n \leq 196$ and when x and y aren't very large. It is fascinating to imagine what she might have achieved if she'd had any support at all.

Image: `https://commons.wikimedia.org/wiki/File:Portrait_Sophie_Germain.jpg`

Kurt Gödel was an excellent student. His brother wrote, "At the time it was rumored that in the whole of his time at High School not only was his work in Latin always given the top marks but that he had made not a single grammatical error." While still in high school, he was studying math at the university level. He earned a Ph.D. from the University of Vienna, studying formal logic and the philosophy of mathematics, and was hired to teach there.

Kurt Gödel

At the time, the world of mathematics was reacting to a crisis: the unexpected and shockingly unintuitive discoveries of the 1800s, such as non-euclidean geometry, and the existence of pathological examples in analysis and topology, had sparked an urgent need for unambiguous language and rigorously logical reasoning. Hilbert was modernizing Euclid's axiomatic development of geometry, while Dedekind and others rethought the definitions of numbers themselves. The grandest project was a redesign of the foundations of mathematics itself, building upon a carefully designed collection of axioms for set theory (the "Zermelo-Fraenkel axioms, with the axiom of choice"). Towards this goal, Bertrand Russell and Alfred North Whitehead had published (between 1910 and 1913) the massive *Principia Mathematica*, intended to establish a trustworthy starting point from which all mathematics could proceed.

After earning his doctorate, Gödel established limits on the aspirations of all of these projects at once by publishing "On formally undecidable propositions of *Principia Mathematica* and related systems," in which he proved his famous Incompleteness Theorems. The first of these states that any axiomatic system rich enough to fully explain integer arithmetic will always contain statements about that subject that are true but cannot be proved; the second is that no axiomatic system can be proved to be free from contradictions by valid reasoning within that system. There was, in other words, no longer any hope of organizing mathematics into a single consistent and complete presentation.

Throughout his life, Gödel was constantly concerned about his health. At the age of six, he had suffered through a case of rheumatic fever. He recovered well but, reading medical books while still a child, he learned that rheumatic fever could damage the heart, and he became convinced that this had happened to him. He was also seriously disturbed when one of his former professors was killed by a Nazi student. For both of these reasons, when World War II began, Gödel was unwilling to be drafted into the German army, and emigrated to the United States. Moving to Princeton, he became friends with Albert Einstein, and published some work on relativistic physics, as well as more results about axiomatic systems. Eventually, his medical anxiety developed into a conviction that he was being poisoned, and in his efforts to avoid this, he died of malnutrition in 1978.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Godel`

Édouard Goursat lived the ordinary life of a modern professional mathematician, earning his Ph.D. in 1881 from the prestigious École Normale Supérieure in Paris, then working as a professor at the University of Toulouse, the ENS, and the University of Paris. (His doctoral advisor, Gaston Darboux, is known for defining the upper and lower Riemann sums, and using them to define what it means for a real function to be integrable.)

Édouard Goursat

Soon after defending his thesis, Goursat published his "Proof of Cauchy's theorem," showing that if a complex function $f(z)$ is differentiable on, inside, and near a simple closed curve C, then $\int_C f(z)\,dz = 0$. Cauchy's original proof was based on the additional assumption that $f'(z)$ was continuous, but Goursat's proof didn't use that assumption. This was important: he could now show that such a function f automatically has a continuous derivative, and is in fact infinitely differentiable! Thus Cauchy's integral theorem, with the extra hypothesis removed, is now often called the Cauchy-Goursat theorem.

Like many other mathematicians of the time, Goursat was interested in the geometry of spaces in four or more dimensions. He was the first to classify all the ways that reflections of four-dimensional space could generate a finite group of linear transformations, and he studied the various smooth surfaces in three dimensions that have the same planes of symmetry as the Platonic solids. At this time, he also proved what we now call "Goursat's lemma," (or, amusingly, "Goursat's other theorem") which classifies the subgroups of a direct product $A \times B$ of groups A and B.

Goursat studied what are now called differential forms. A differential form is basically the type of expression that comes after an integral sign: a function times an expression involving differentials. In this language, Goursat observed that he could write Green's theorem, Stokes's theorem, and the divergence theorem in multivariable calculus in the unified form $\displaystyle\int_{\partial S} \omega = \int_S d\omega$. (The theorems had already been combined into a single "generalized Stokes's theorem" by the Italian mathematician Vito Volterra.)

After the turn of the century, Goursat focused on writing a new textbook, *Cours d'analyse mathématique*, covering calculus and differential equations. In this book, which was well received, he named "l'Hôpital's rule" after the Marquis de l'Hôpital for the first time. The text was translated into English and used in many American universities.

Goursat was well respected in his lifetime, winning many prizes for his mathematical work, and even received the Legion of Honor, the highest decoration awarded in France.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Goursat/`

Evelyn Boyd Granville

Born in Washington D.C. in 1924, Evelyn Granville (*née* Boyd) experienced the Great Depression as a young child. She was encouraged in her education by her aunt, and came to regard school as a critical source of opportunity — especially, she wrote, because she was Black:

> We accepted education as the means to rise above the limitations that a prejudiced society endeavored to place upon us.

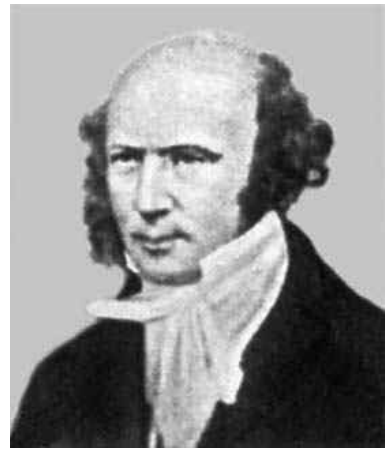
Granville graduated with honors from Smith College, where she focused on math and astronomy and worked for the National Bureau of Standards during the summers. She earned her Ph.D. at Yale in 1949, working on functional analysis — one of the first three African-American women to earn a doctorate in mathematics. She took a temporary position at New York University, where she studied differential equations, then became a professor at Fisk University in Tennessee. While there, she and three colleagues attended the annual meeting of the Southeastern Section of the Mathematical Association of America, but because they were Black, they were barred from the official banquet and the speech given there by the MAA's national president. This shameful act of discrimination, once publicized, forced the MAA to formulate its first anti-discrimination policy.

In Tennessee, Granville was confronted with enough obstacles, due to discrimination based on both her race and her gender, that she left in 1952 for a full-time job back at the National Bureau of Standards. This was the beginning of the age of computing, and Granville's work (studying the design of fuses for missiles) brought her into contact with mathematicians who worked as computer programmers. By the end of 1955, she had transitioned into a job with IBM Corporation, and ended up working on an team contracted to write software for NASA. Her software helped NASA prepare for the Project Mercury missions by analyzing the orbits of satellites, and she was also involved in the Apollo program. Granville wrote:

> I can say without a doubt that this was the most interesting job of my lifetime - to be a member of a group responsible for writing computer programs to track the paths of vehicles in space.

Granville returned to teaching in 1967, and taught computer programming and mathematics, including math courses for future teachers, until her retirement in 1997.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Granville/`



William Rowan Hamilton

William Rowan Hamilton was such an exceptional student that, before he graduated from Trinity College in Dublin, they appointed him as a Professor of Astronomy, in a position that technically made him the "Royal Astronomer of Ireland." Though he had studied quite a bit of mathematical work on both optics and dynamics (the study of motion), he never spent much time actually observing the sky, and his interest was entirely in pure mathematics.

At the beginning of the 1800s, mathematicians were looking for a way to make imaginary numbers respectable. In the previous century, Euler had used the symbol $i = \sqrt{-1}$, but without giving a clear definition of what this could possibly mean. Following geometric work of Wessel (1797, Denmark) and Argand (1806, France), Hamilton showed in 1831 that multiplying ordered pairs of real numbers according to the formula $(a, b)(c, d) = (ac - bd, ad + bc)$ made it possible to define $i = (0, 1)$. This concrete representation of $\sqrt{-1}$ not only put an end to the controversy about the existence of imaginary numbers, but provided an extremely useful way to visualize them.

After this, Hamilton spent years looking for a way of multiplying ordered triples of real numbers that had good algebraic properties, but couldn't find one. (It's now known that this problem is impossible, unless you don't insist on very good properties.) But one day in October 1843, on a walk with his wife, Hamilton suddenly realized that he could define a useful product in four dimensions, and celebrated by carving the necessary formulas in the stonework on the Broome Bridge: $i^2 = j^2 = k^2 = ijk = -1$. He spent the rest of his life studying the numbers $a + bi + cj + dk$, which he called "quaternions."

Later, once people started using vectors, it became common to represent the quaternion $a + bi + cj + dk$ by the expression $(a, \vec{v})$, where $\vec{v}$ stands for the vector $\langle b, c, d \rangle$. In this notation, the product of quaternions leads to both the dot product and the cross product (neither of which was known at the time): we have $(0, \vec{v})(0, \vec{w}) = (-\vec{v} \cdot \vec{w}, \vec{v} \times \vec{w})$.

Hamilton was a friend of the poet William Wordsworth, and wrote a lot of poetry himself, usually when he was unhappy. He was often unhappy, partly because he spent his whole life in love with a woman who was married to an older, richer man.

Hamilton thought deeply about the law of conservation of energy. To apply this law, he developed the functions that are now called Hamiltonians, which give the total energy of a physical system over time in terms of the momentum and position of the particles in the system. Hamiltonian functions, suitably adapted, survived the revolution in physics in the early 1900s, and today they are a critical part of every physicist's toolbox.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Hamilton/`

David Hilbert

In 1900, David Hilbert was the most influential mathematician in the world. He was a professor at the University of Göttingen in Germany, in the mathematics department where Gauss and Riemann had taught. He had proved a revolutionary new result in the field of abstract algebra, called the Hilbert basis theorem. And he had just published *Grundlagen der Geometrie (Basics of Geometry)*. Here, he provided a new collection of twenty-one axioms for Euclidean solid geometry (one of which later turned out to be redundant), addressing the instances where the *Elements* didn't satisfy modern standards of rigor. This axiomatic system carried extra significance in the wake of nineteenth-century developments in non-Euclidean geometry, as mathematicians and scientists were forced to pay extra attention to the logical foundations of their areas of study. Hilbert's axioms formed a new model for mathematical practice, and more and more disciplines were formalized.

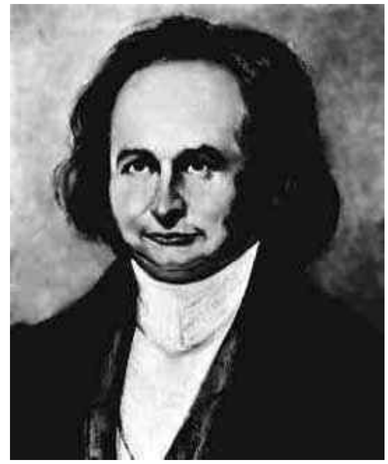
Hilbert gave a speech in Paris in 1900 whose importance cannot be overstated. Speaking to the International Congress of Mathematicians, he listed twenty-three unsolved problems which he considered top priorities for mathematical research in the twentieth century. In fact, a lot of the mathematics community fell in with this plan: the "Hilbert problems" are well known worldwide, and solving one of them is among the most prestigious accomplishments that a research mathematician can dream of. Over half of the problems have been solved completely, and progress has been made on all of them. In a few cases, like the first problem (the "continuum hypothesis"), it is now known that no solution is possible (which is, in its own way, a solution).

A century later, the Clay Mathematics Institute tried, fairly successfully, to duplicate Hilbert's achievement, and identified seven "Millennium problems" as challenges for the twenty-first century. Of these, the Riemann conjecture is Hilbert's eighth problem, and the Birch and Swinnerton-Dyer conjecture is a refinement of his tenth problem. While Hilbert had to rely on his reputation and persuasive ability to attract people to his problems, the Clay Institute was able to take a different approach: they pledged a $1 millon prize for each of the solvers. However, when Grigori Perelman proved the Poincaré conjecture in 2010, he turned down the money. The other six prizes are still on the table!

Hilbert is buried in Göttingen, under a tombstone that quotes the last six words of a speech he gave in Königsberg in 1930:

Wir müssen wissen, wir werden wissen. (We must know; we shall know.)

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Hilbert/`



Carl Gustav Jacob Jacobi

Carl Jacobi came from a wealthy family, and was a stunningly good student. By the age of 12, he had the qualifications to enroll at the University of Berlin, but they wouldn't let him in until he was 16. He spent the time doing advanced study on his own, working through Euler's famous calculus text *Introductio in analysin infinitorum,* among other things. The result was that by the time he got to college, he had learned most of what they could teach him, and ended up doing self-study there too. By the time he was 21, he earned his doctorate, and got a job as a professor (though, in 19^{th}-century Prussia, he had to convert from Judaism to Christianity before he could be hired).

Just over two hundred years earlier, Bachet had conjectured that every positive integer could be written as a sum of four perfect squares: for example, $168 = 10^2 + 8^2 + 2^2 + 0^2$. Lagrange proved this conjecture in 1770, but Jacobi went further: he showed in 1834 that n can be written as a sum of squares in k different ways, where k is eight times the sum of the divisors of n that aren't divisible by 4. Specifically, there are $8(1 + 2 + 3 + 6 + 7 + 14 + 21 + 42) = 768$ different ways to write $n = 168$ as a sum of four squares, of which 192 can be found by shuffling the four terms of $10^2 + 8^2 + 2^2 + 0^2$ and/or changing the signs on the 10, 8, and 2.

Jacobi proved this and other facts in number theory by working with elliptic functions. The "Jacobi elliptic functions" are a family of functions based on the "Jacobi amplitude," which is the inverse function of $f(x) = \displaystyle\int_0^x \frac{dt}{\sqrt{1 - m\sin^2 t}}$. Jacobi and his Norwegian rival Abel first suggested looking at such inverse functions, and it was extremely productive.

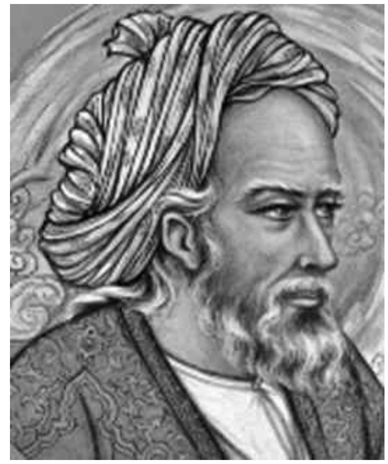
Calculus students know Jacobi's name because of the Jacobian matrix, but Jacobi never saw one himself! Jacobi died in 1851, the word "matrix" was first used in mathematics by J. J. Sylvester in 1850, and the operation of matrix multiplication comes from two papers published by Arthur Cayley in 1855 and 1858. However, determinants were (strangely) known much earlier, and it was Jacobi who introduced the Jacobian determinant for the first time, and developed a clear theory of determinants. He was also influential in promoting the $\frac{\partial f}{\partial x}$ notation for partial derivatives, which was first used by Legendre.

Jacobi was a hard worker, writing to a friend, "Certainly I have sometimes endangered my health by overwork, but what of it? Only cabbages have no nerves, no worries. And what do they get out of their perfect wellbeing?"* But it did eventually cause health problems, and he travelled to Italy to recuperate in the milder weather, only to get disastrously involved in politics and lose the support of his patron, the King of Prussia.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Jacobi/`

*Quoted in Eric Temple Bell, *Men of Mathematics* 329–330.

Ghiyāth al-Dīn Abū al-Fatḥ ‘Umar ibn Ibrāhīm Al-Nīshābūrī al-Khayyam studied mathematics and astronomy in his hometown of Nishapur, a large city on the Silk Road in the modern nation of Iran. As a young adult he joined the court of Sultan Malik-Shah I of the Seljuk Empire, and was charged with opening an observatory and reforming the calendar. With his fellow scientists, he was able to measure the length of a year, finding a value that was accurate within a few millionths of a day. Based on this computation, they invented the Jalālī calendar, still used in Iran in a simplified form. The calendar includes 8 leap days every 33 years, and corresponds to Earth's orbit around the sun more precisely than the Gregorian calendar that would be developed in Europe 500 years later.



Omar Khayyam

In an early work, Khayyam solved the equation $x^3 + 200x = 20x^2 + 2000$ by a geometric construction, and before long he managed to classify all cubic equations that could be solved geometrically by intersecting conic sections. He was the first to show that such cubic equations can have more than one solution, and the first to imagine a general cubic formula, which would not be found until the 1500s. In a work that no longer exists, he also showed how to calculate n^{th} roots using binomial coefficients — the numbers that form what we call “Pascal's triangle,” although Blaise Pascal was born in 1623.

Khayyam also took part in the long struggle to improve on the foundations of Euclidean geometry. In his *Commentary on the Difficulties Concerning the Postulates of Euclid's Elements*, he identified errors and hidden assumptions in previous attempts to prove Euclid's parallel postulate, and in his own effort to do so he was the first to discover the properties of what are now called Saccheri quadrilaterals in non-Euclidean geometry (named after a man born in 1667). In the same book, he took important steps toward the modern concept of real numbers by suggesting that it might be possible to consider ratios to be numbers, and by showing that the different definitions of equality of ratios given by Euclid and by Islamic scholars like al-Mahani were actually equivalent.

Despite the excellence of his mathematics, Khayyam may now be better known for a collection of quatrains (four-line poems) attributed to him, which were translated into English by Edward FitzGerald under the title *The Rubaiyat of Omar Khayyam*. These include the famous verses, “The Moving Finger writes; and, having writ, / Moves on: nor all your Piety nor Wit / Shall lure it back to cancel half a Line, / Nor all your Tears wash out a Word of it,” as well as “A Book of Verses underneath the Bough, / A Jug of Wine, a Loaf of Bread — and Thou / Beside me singing in the Wilderness — / Oh, Wilderness were Paradise enow!*”

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Khayyam/`

* “Enough.”

Felix Klein

Felix Klein earned his Ph.D. from the University of Bonn at the age of 19. He had wanted to learn physics, but his doctoral advisor, Julius Plücker, had shifted to working on geometry. Within a few years, he became a professor at the Friedrich-Alexander University in Erlangen, and it was there that he developed what we now call the "Erlangen program."

At the time, geometry was in turmoil, as Bolyai, Gauss, and Lobachevsky had found consistent "non-Euclidean" versions of geometry, in which Euclid's parallel postulate was violated. Klein showed how to understand these different discoveries within a single framework, by insisting that geometry should be studied in terms of transformations. Euclidean geometry was to be carried out using "isometries," like reflections and rotations: transformations of the plane that preserve distances. He also gave a usable model for the projective plane and its group of symmetries, and argued that the symmetries of a space determine the properties that are worth studying in that space.

In fact, Klein was able to design a method of constructing a space so that Euclidean geometry and non-Euclidean geometry occurred as special cases. This allowed him to prove that non-Euclidean geometry couldn't involve any contradictions, unless Euclidean geometry was also inconsistent. The standard viewpoint, that Euclid had described the only possible shape for the physical universe, was overthrown in favor of the modern philosophy that in any given context we should assume the geometric axioms that are most convenient.

Two famous objects familiar to math students are named in honor of Klein. The "Klein four-group" is the non-cyclic group with four elements: for example, the group of symmetries of a non-square rectangle. The "Klein bottle" is the most famous non-orientable surface, formed (in a space of at least four dimensions) by connecting the ends of a cylinder in the orientation that doesn't yield a torus.

Klein spent the second half of his life in Göttingen. The university's reputation was already well established: it had been home to many famous mathematicians, including Gauss. However, Klein was able to develop it into the world's premier center of research in mathematics and physics, bringing in Hilbert as the centerpiece. He also worked to make the University admit women as students. He built up the *Mathematische Annalen* (Mathematical Annals) until it was the most prestigious of all mathematical journals. He was also interested in math education, and may be the single person most responsible for making the concept of "function" part of a standard high-school education.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Klein/`

Sofia ("Sonia") Kovalevskaya grew up in a wealthy family. Her father was a general in the Russian military, and the heir to a large estate, so she grew up in a manor house, educated by private tutors. Among the family's friends was the author Fyodor Dostoevsky, who wanted to marry Sofia's older sister.

Sofia Kovalevskaya

Her family redecorated, but they didn't order quite enough wallpaper. Since the wallpaper had been ordered from far-off Saint Petersburg, they couldn't get another roll very easily, so the walls of Sofia's bedroom were covered with the paper that was handy: the lecture notes from the calculus classes her father had taken while training to become an artillery officer. She got so interested in algebra that her father got alarmed and fired her tutor; she managed to borrow an algebra book and hid it under her pillow, studying after dark.

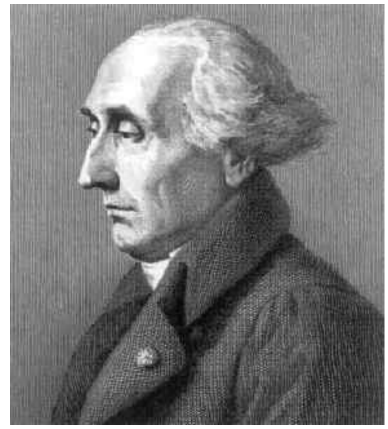
By chance, one of the family's neighbors wrote a physics textbook, and gave Sofia's father a copy. In order to understand the book, she had to teach herself trigonometry. When the neighbor found out, he pressed her father to let her develop her mathematical talent, but he resisted. Ultimately, at eighteen, she found a solution: she married, and (as was legally required then in Russia) got her husband's permission to go to school, first at the University of Saint Petersburg, then in Heidelberg (where Kirchhoff was one of her lecturers), and finally in Berlin. Neither German university would let Kovalevskaya enroll as a student, but in Berlin she was able to study privately with the great Karl Weierstrass.

In her three years in Berlin, Kovalevskaya earned her doctorate, and wrote a world-class paper on partial differential equations, but being a woman, she could not get a job as a professor. She and her husband tried to apply their scientific ability to investing, but lost everything. They separated, and financial troubles eventually drove him to suicide.

Finally, with help from the famous Swedish mathematician Gösta Mittag-Leffler, Kovalevskaya won the chance to teach, unpaid, for a year at the University of Stockholm. Once she was in the door, she earned a chair as a professor, published new research, became an editor of a journal, and won prestigious prizes from the French and Swedish academies of Science. Now, according to biographer A. H. Koblitz, "Kovalevskaia was indeed considered the equal of anyone of her generation." Tragically, it didn't last: she got sick while traveling and died of pneumonia shortly after her 41^{st} birthday.

Many universities now honor Sofia's memory by celebrating "Sonia Kovalevsky Day" with activities to encourage young women to get involved in math and other STEM subjects.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Kovalevskaya/`



Joseph-Louis Lagrange

Lagrange's interest in mathematics was inspired by a good teacher at his hometown college in Turin, Italy. He took an interest in the "tautochrone problem," which had been solved in 1659 by Huygens. The goal was to find a curve with the property that a frictionless mass, initially at rest, always takes the same amount of time to slide down the curve to its lowest point, no matter where on the curve the mass starts out. Huygens had shown that cycloids had this property. Lagrange gave his own proof of this fact at the age of 18, and sent his work to Leonhard Euler. Euler, who was at the time the most influential mathematician in the world, wrote back right away to say that he was impressed!

Within a year, Lagrange had a job as a mathematics professor in Turin, though he was never a good lecturer. He spent his life working on mathematical analysis of physics, and in *Mécanique Analytique* (1788), he presented the approach that is now called Lagrangian mechanics, in which energy, rather than force, is the primary concept. The Lagrangian function gives the difference between the kinetic and potential energy of a mass, and its integral with respect to time is the particle's "action." In many contexts, it turned out that particles moved in such a way that their action was minimized, or at least constant. Thus Lagrange was interested in finding minimum values of multivariable functions, subject to various constraints, and he developed the method of Lagrange multipliers to attack this problem.

Lagrange spent years studying astronomy, and in his investigation of the orbits of planets and comets, he discovered how to solve linear differential equations by "variation of parameters," a method which is still taught in differential equations courses. He also did pure math: In 1770, he proved that every natural number is a sum of four perfect squares. He also analyzed why the roots of polynomials with degree 4 or less could be found using formulas. Along the way, he proved that the number of polynomials that can be formed by permuting the variables in a polynomial function is always a factor of $n!$, where n is the number of variables that appear in the polynomial. After Lagrange's death, when later mathematicians defined the concept of a "group," they rephrased and extended this result to prove what is now called Lagrange's theorem: The number of elements in a finite group is always a multiple of the number of elements in any subgroup it contains.

From Turin, Lagrange moved to Berlin, and then to Paris, where he lived during the French Revolution and the subsequent Reign of Terror. Having been born in Italy, he was subject to arrest, and the loss of all his property. It was Antoine Lavoisier, the greatest chemist of the era, who saved Lagrange from this fate (though he could not save himself, and was guillotined in 1794). Lagrange survived into the following regime, and received many awards, including the Legion of Honor, from Napoleon.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Lagrange/`

Pierre-Simon Laplace's father sent him to school to become a priest, but Laplace did so well in his math classes that he dropped out and moved to Paris to learn from the very influential mathematician Jean d'Alembert. He quickly settled in as a professor at the École Militaire (Military School).

Pierre-Simon Laplace

While studying the orbits of planets, Laplace introduced the operator ∇^2, the "Laplacian," which is the divergence of the gradient of a multivariable function. He showed that gravitational potential energy satisfies the equation $\nabla^2 u = 0$, which is now called Laplace's equation. Solutions of Laplace's equation, called harmonic functions, also occur in many other applications.

In 1809, in a "Memoir on approximations of formulas that are functions of very large numbers," Laplace solved a differential equation by assuming that the solution had the form $y = \int e^{-sx} f(x)\,dx$. Similar ideas had occurred previously to Euler and Lagrange, but it was Laplace who developed them into a method. Still, the "Laplace transform" wasn't fully developed in its present form until the 1930s and 1940s.

Throughout his career, Laplace studied probability, and in his book *Analytic Theory of Probability* he brought together many fundamental techniques for computing and working with probability and statistics. In this book, he proved a special case of the famous central limit theorem, showing that errors in large sets of astronomical observations would follow a normal distribution ("bell curve").

Considering his extensive work on probability, it seems odd that Laplace became famous for his belief in determinism — that the future could be predicted perfectly given the exact position and velocity of every particle in the universe. Of course, he didn't claim that humans could accomplish this: it would require a superintellect, sometimes called "Laplace's demon." When Napoleon challenged him on trying to explain astronomy without mentioning God, Laplace boldly replied, "I have no need of that hypothesis."

Laplace was a great mathematician, but he was so arrogant about his skills that it damaged his relationships with other scientists. It didn't help that he came through the political upheaval of his time — the French Revolution, the Reign of Terror, Napoleon's empire, and the restoration of the monarchy — by willingly supporting whoever was in power at any given moment. He was briefly Minister of the Interior for Napoleon, but was fired after six weeks, according to Napoleon, "because he brought the spirit of the infinitely small into government." Still, Napoleon made Laplace a count, and King Louis XVIII later elevated him to the rank of marquis.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Laplace/`

Pierre Laurent

Pierre Laurent was born in Paris, just as France was losing the Napoleonic Wars. He was less than a year old when his family moved to England, where his mother had been born, and he didn't return to France for about ten years. He went to college in Paris, then joined the military as an engineer and took part in the French invasion of Algeria. When he returned to France, he worked on improving the port at Le Havre, which became the primary port in France.

During his time at Le Havre, Laurent began to work on original research in mathematics. In 1842, when the French Académie des Sciences published the topic for their annual Grand Prix competition, Laurent sent in a paper. Unfortunately, he submitted it after the deadline, so it couldn't win the prize, but it was still considered for publication. One of the referees was the older, very influential mathematician Augustin-Louis Cauchy (who had also served as a military engineer). Cauchy took the opportunity to present his own proof of Laurent's main theorem, based on a paper Cauchy had published in 1837. Though Cauchy gave a positive review, his attempt to take credit for Laurent's idea kept the paper from being published.

Since the time of Newton, it has been a standard technique to represent functions by Taylor series. Laurent saw that it could be productive to allow negative exponents, especially when dealing with complex functions. For example, the function $f(z) = \dfrac{1}{z^2+1}$ can be written, using the geometric series formula, as the Taylor series $f(z) = \displaystyle\sum_{n=0}^{\infty}(-1)^n z^{2n}$, but this only converges when $|z| < 1$. When $|z| > 1$, though, we can use creative algebra to write

$$f(z) = \frac{1}{z^2} \cdot \frac{1}{1+z^{-2}} = \frac{1}{z^2} \sum_{n=0}^{\infty} \left(-z^{-2}\right)^n = z^{-2} - z^{-4} + z^{-6} - z^{-8} + \cdots = \sum_{n=-\infty}^{-1} (-1)^{n+1} z^{2n}.$$

Just like a Taylor series, this "Laurent series" can be integrated and differentiated as if it were a polynomial, making it much easier to work with than the original formula.

Remarkably, Laurent wrote two more papers that were reviewed by Cauchy; Cauchy recommended both for publication, and both were rejected. Moreover, Cauchy nominated Laurent to the Académie des Sciences, but he was not appointed. Laurent also wrote about the theory of wave motion, applied to light waves and sound waves as well as to waves in liquid, and once again, that work involved him in controversy with Cauchy, who had taken a different approach to similar problems.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Laurent/`

Gottfried Wilhelm von Leibniz

Gottfried Leibniz is famous as the co-creator of calculus (along with Isaac Newton, who despised him), but he was not originally motivated by mathematics itself. His true goal was to organize all human knowledge into a single organized system, and especially to create a perfect language, the "universal characteristic," of which he wrote that "the symbols and even the words would direct the reason; and errors, except those of fact, would be mere mistakes in calculation."

However, Leibniz saw that this project would have to involve more mathematics than he (a philosopher with a doctorate in law) understood, and he turned to the physicist Christian Huygens for lessons. By 1675, Leibniz had surpassed Huygens, proving the fundamental theorem of calculus. Newton had done the same nine years earlier, but had not published. When Leibniz announced his own discoveries, Newton accused him of plagiarism, and the two were never reconciled, though Leibniz honored Newton's work, saying, "Taking mathematics from the beginning of the world to the time of Newton, what he has done is much the better half." But it was Leibniz, not Newton, who created the integral sign $\int$ and the differential notation dx that we still use today. He was also the first to use the dot $\cdot$ for multiplication, writing to Johann Bernoulli, "I do not like $\times$ as a symbol for multiplication, as it is easily confounded with x. . . ."

The concept of binary arithmetic, which is so fundamental to our computerized world that it is hard to imagine life without it, was first proposed and developed by Leibniz. He worked with Bernoulli on logarithms of negative numbers, a project that would eventually be resolved by Euler and perfected by Riemann. He also designed, and demonstrated a partially complete prototype of, a mechanical calculator which could carry out addition, subtraction, multiplication, division, and square roots.

Beyond the world of mathematics, Leibniz worked as a diplomat working for peace in the court of the King of France. He led mining projects, as part of which he was the first to suggest that the Earth had hardened from an initial molten form, and a tireless correspondent who was in constant scientific dialogue with hundreds of thinkers across Europe. In his primary contribution to philosophy, Leibniz argued that everything in the universe was made of "monads," each of which had a form of consciousness, constantly perceiving and being perceiving by all of the others. The simplest of these, he said, was God, of whom all other monads are imitations. Since all existence imitates perfection, he claimed, we live in the best of all possible worlds, which explains why bad things can happen even under the control of an all-powerful and perfectly good God. Some aspects of life make this theory difficult to accept fully, and the foolish Dr. Pangloss in Voltaire's *Candide* is a parody of Leibniz and his followers.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Leibniz/`

As a sixteen-year-old, Joseph Liouville entered the prestigious École Polytechnique in Paris. He took a calculus class taught by André-Marie Ampère, after whom the SI unit of electrical current is now named. Cauchy, maybe then the world's most influential mathematician, also taught there at the time, so Liouville was well positioned to learn about the newest developments.

Joseph Liouville

After graduating, he became a transportation engineer, but his heart wasn't in it. Instead, starting in 1831, he took jobs at a number of Paris schools at the same time, including the École Polytechnique. While teaching an incredibly heavy load of courses, he studied the conditions under which an algebraic function (one with a formula expressible in terms of arithmetic, powers, roots, exponentials, and logarithms) has an algebraic antiderivative. This work was later used to prove that the antiderivatives of many functions, such as $f(x) = e^{x^2}$, can't be expressed by any finite formula.

Liouville also spent time trying to prove that e is transcendental; that is, that no polynomial with integer coefficients has e as a root. Though he didn't succeed (Charles Hermite was the first to figure it out, in 1873), he did manage to give the first explicit example of a transcendental number, which is now called Liouville's number:

$$L = \sum_{k=1}^{\infty} 10^{-k!} = 0.1100010000\ldots,$$

with 1's in the decimal places indexed by factorials, and 0's in all other places. To do this, he showed that, unexpectedly, transcendental numbers can be generally approximated more closely by rational numbers with small denominators than algebraic numbers can.

By 1838, he had earned a full-time job as a professor at the École Polytechnique. While there, he proved, and announced in his lectures, that the constant functions are the only bounded functions that are differentiable on the whole complex plane; this fact is now called "Liouville's theorem," though Cauchy seems to have proved it first.

The beginning of Liouville's career coincides with the French Revolution of 1830, as seen at the climax of Victor Hugo's *Les Miserables.* In 1848, King Louis-Philippe was dethroned, and Liouville was elected to the body that wrote the constitution for the new Second Republic. After several months, they established a democratic government under a directly elected President, and the people responded by electing Emperor Napoléon's nephew, who before the end of his four-year term had used claims of a communist conspiracy to dissolve the National Assembly and proclaim himself Emperor Napoléon III. Even before this, Liouville had run for election to the Assembly and lost, and from then on he was depressed and bitter.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Liouville/`

Colin Maclaurin

The young Colin Maclaurin was such a good student that he was admitted to the University of Glasgow at the age of eleven. After studying a broad range of topics, he presented a thesis on Newton's theory of gravity, graduated, and started studying to become a minister. He gave that up after a year, though, and went home to study math independently.

By the age of 19, he won a job as a professor at the University of Aberdeen. While there, he had the opportunity to meet with Newton, and must have made a strong impression, because he was elected to the Royal Society, a distinguished group of England's leading scientists. He also left for two years to travel around Europe with a diplomat's son, without informing his school. Remarkably, he got his job back when he returned! He was an excellent teacher: Alexander Carlyle, who led the Church of Scotland, later wrote:

> Maclaurin was at this time a favourite professor, and no wonder, as he was the clearest and most agreeable lecturer on that abstract science that ever I heard.

After eight years, Maclaurin moved to the University of Edinburgh, assisted by a strong recommendation from Newton, and taught there for the rest of his life. He helped to create the theory of actuarial studies, and his work was applied by the Church of Scotland in their charitable planning. His major achievement, though, was the publication in 1742 of his *Treatise of fluxions*, an organized and unified presentation of Newton's calculus. A major goal was to clarify the logical justification of the subject, which had come under criticism; another was to show how the subject could be applied.

The Maclaurin series, the power series that converges to an arbitrary smooth function $f(x)$ on an interval $(-R, R)$, appears in the *Treatise of fluxions*, though Brook Taylor published the idea over 25 years earlier. In fact, apparently James Gregory made the discovery almost 50 years before that, but didn't realize it was new, and didn't publish it. Also in the *Treatise*, Maclaurin was the first to state and prove the integral test for convergence or divergence of a numerical series.

Charles Stuart ("Bonnie Prince Charlie"), whose grandfather, King James II of England, lost his throne in the Glorious Revolution of 1688, raised a rebellion in Scotland in 1745. Maclaurin had a leading role in preparing to defend Edinburgh against the revolutionaries, and had to flee to England when the city surrendered. His travels in cold winter weather surely played a role in the illness that eventually took his life.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Maclaurin/`

Isaac Newton attended Cambridge University, planning to study law. While there, in 1663, he bought a book on astrology, but couldn't understand it. This motivated him to learn mathematics, and he dove into the work of Euclid and Descartes, among others. Just after he earned his bachelor's degree, a plague forced Cambridge to close, and Newton had to go home. It was in 1666, during this enforced isolation, that Newton first recorded the fundamental theorem of calculus:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Isaac Newton

If Newton could see that equation, he would hate it, because it's written in the notation invented by his rival, the German philosopher Gottfried Wilhelm von Leibniz. The endlessly quarrelsome Newton was certain, mistakenly, that Leibniz had stolen his ideas, and was only pretending to have discovered them himself. Newton's hostility was bitter enough that it seriously disrupted collaboration between mathematicians in Britain and those in continental Europe for over a hundred years after his death.

When Leibniz published his own work on calculus, it forced Newton to do the same. This took him until 1687, when he released the *Principia (Mathematical Principles of Natural Philosophy)*. This massive text first introduces some calculus, then develops the universal theory of gravity, showing that Kepler's laws of planetary motion follow from a gravitational force between the sun and the planets that varies inversely with the square of the distance between them. Besides its importance in physics, Newton's theory of gravity was a major shift in the scientific method: he argued that it was good enough to predict the effect of the gravitational force between two heavenly bodies, even without explaining how that force was transmitted across the space between them.

Newton also published a major text called *Opticks*, in which he studied the properties of light, and showed for the first time that white light is composed of a spectrum of different colors, which can be separated by a prism. He realized that light wasn't the same thing as our perception of it: among other experiments, he inserted a blunt needle between his eye and eye socket, recording the patterns he saw as he deformed his own eyeball.

Newton was a devoted alchemist, persistently seeking the famous "philosopher's stone" to turn other metals to gold, and he searched the Bible intently for secret messages about the end of the world. In 1696, he took a job running the Royal Mint, which left him little time to do new science. However, when Johann Bernoulli set two mathematical problems as a challenge to the scientific community, Newton answered both in a single day. Though he published his solution anonymously, its brilliance gave away his identity. Bernoulli wrote, "tanquam ex ungue leonem." ("We know the lion by his claw.")

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Newton/`

Emmy Noether

Emmy Noether's father was a professor of mathematics in the German university city of Erlangen. Initially, she studied foreign languages, and earned a certificate as a qualified teacher of English and French. However, she changed her mind and began to study mathematics. This was a challenge: women were not allowed to enroll in German colleges, and she had to get permission from individual professors to attend their lectures. When this rule changed in 1904, the 22-year-old Noether officially registered at Erlangen, earning her doctorate by 1907.

At the time, Noether could not be hired to teach because she was a woman, but she began to publish papers, and even supervised doctoral students, using her father's name on the official paperwork. At first, Noether specialized in the study of invariant polynomials: polynomials in multiple variables whose formulas are symmetric in various ways. A simple example is $f(x, y) = x^2y + xy^2$, which doesn't change when the variables x and y are interchanged. By working with linear combinations of polynomials, she developed a deep understanding of mathematical structures that eventually reshaped abstract algebra.

In 1915 Noether was invited to move to Göttingen, where Hilbert and Klein were working on the brand-new theory of relativity and needed an invariant theory specialist. She quickly proved what physicists call Noether's theorem: there is a conservation law for every continuous symmetry in a physical system. She found ways to be productive even while facing persistent gender bias. During the struggle to hire her officially to Göttingen's faculty, Noether taught her own courses under Hilbert's name. Even later in her career, many of her results were published under the names of her male collaborators.

Once she became a professor in her own right, she began to study the structure of ideals in ring theory. It was Noether who gave the modern abstract definition of an integral domain in her important 1921 paper "Ideal theory in rings." Famously, she and Hilbert showed that many properties of polynomial rings follow from a single axiom: that in a given ring there are no infinitely long sequences of nested ideals. Rings satisfying this axiom are now called Noetherian.

Despite her growing fame, Noether, who was Jewish, lost her job in 1933 due to pressure from the Nazi Party. It was unsafe for her to stay in Germany, and she left for a temporary position at Bryn Mawr College in Pennsylvania, which, as a school that only admits women, would not discriminate against her on the basis of gender.

Sadly, soon after moving to the U.S., Noether was found to have cancer, and died at the age of 53 after surgical complications.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Noether/`

Blaise Pascal

Blaise Pascal was raised in Paris by his father, an amateur mathematician, who home-schooled him, and didn't allow him to study mathematics. Naturally, this prohibition fascinated Pascal, who proved, at the age of twelve, that the sum of the three angles of a triangle equals two right angles. This convinced his father to give him a copy of Euclid's *Elements*, and to let him attend the meetings led by Marin Mersenne. When he was sixteen, he presented some of his own work in projective geometry at one of these meetings. Among these results was "Pascal's mystic hexagram," showing that if a hexagon is inscribed in a conic section, then the three points where opposite sides intersect are collinear.

In the 1640s, to help with his father's work as a tax collector, Pascal invented a mechanical adding machine: the first digital calculator. These could handle the difficulties of French currency (240 deniers in one livre), and were actually manufactured, though not many were sold. It seems fitting, therefore, that one of the first widely-used programming languages is named PASCAL in his honor.

After this, Pascal studied the question of the vacuum in nature. Aristotle had claimed that an empty physical space was impossible ("nature abhors a vacuum"), but Torricelli had done an experiment that suggested otherwise: he turned a glass tube full of mercury upside down in a bowl of mercury, with the opening under the surface, and it didn't empty completely. Instead, a vacuum formed in the top of the tube, with air pressure on the exposed surface holding up the rest of the liquid. Pascal strengthened that experiment: he had his brother-in-law carry the whole apparatus up a high mountain, and watch as the mercury level in the tube dropped. This proof that air has weight is why the SI unit of pressure is called the Pascal.

Gambling was one of Pascal's hobbies, and this led a nobleman to ask why he kept losing at a dice game where he thought the odds would be in his favor. In an effort to answer this question, Pascal established the first mathematical theory of probability in a brief correspondence with Pierre de Fermat. In solving these problems, he made use of what we call "Pascal's triangle," though in China, Yang Hui had written it down before 1300.

After his father died, Pascal wrote *Pensées* (Thoughts), a book of Christian philosophy. Among other things, he argues there that it is rational to believe in God simply on the basis of expected value. His argument, known as "Pascal's wager," has become famous:

> Let us weigh the gain and loss in wagering that God exists. Let us estimate these two chances. If you gain, you gain all; if you lose, you lose nothing.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Pascal/`

Henri Poincaré belonged to a distinguished family: his cousin Raymond was President of France during World War I (though he took office just after Henri's untimely death). As a child, Poincaré was an excellent student in all subjects, but especially mathematics, in which he won national competitions. His math teacher called him a "monster of mathematics," a judgment which would stand the test of time.

Henri Poincaré

Poincaré earned his doctorate at the University of Paris while working on the side as a mining engineer. He became a professor, and his lectures were regarded as somewhat disorganized. His research, too, was sometimes more intuitive than rigorous. However, his knowledge was both broad and deep, and he made major discoveries in multiple fields.

When Poincaré was in his prime, the subject of topology was in its infancy. His 1895 book *Analysis situs* established the structure of that discipline. In particular, he was the first to introduce the ideas of algebraic topology, in which algebraic objects like groups are used to identify the properties of geometric spaces. In 1894, Poincaré defined the fundamental group, in which the elements are equivalence classes of closed paths. He used this and similar objects in an attempt to classify spaces. One of his predictions in that direction became known as the Poincaré conjecture; in 2000 the Clay Mathematics Institute made this one of their seven Millennium Problems, with a million-dollar prize for a proof. (This is the only Millennium Problem that has been solved, and remarkably, Grigori Perelman, the solver, turned down the money.)

Poincaré also deserves credit, with Albert Einstein and Hendrik Lorentz, for the special theory of relativity. He was one of the pioneers in showing that non-Euclidean geometry was consistent, and his model of projective geometry, known as the Poincaré disk model, is still one of the most natural ways to describe what it could mean for a plane to be non-Euclidean. Also, in his important work on the three-body problem, he was the first to identify the phenomena that developed into chaos theory.

It happens that in 1897 a psychologist named Édouard Toulouse studied Poincaré's creative process and recorded his findings in a book.Toulouse wrote, "He will normally start [writing a paper] without knowing where it will end. ... [T]he work seems to lead him on without him making a wilful effort." Describing his own attitude about his studies, Poincaré wrote, "The scientist does not study nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful. If nature were not beautiful, it would not be worth knowing, and if nature were not worth knowing, life would not be worth living."

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Poincare/`

Srinivasa Ramanujan

In 1903, in a medium-sized city in India, the high school student Srinivasa Ramanujan was handed a library book: *A Synopsis of Elementary Results in Pure and Applied Mathematics* by G.S. Carr. It contained thousands of formulas and theorems without proofs, and Ramanujan, after studying its contents, began to expand on them. As he started college, though, he spent all of his time on math, neglecting his other schoolwork, and lost his scholarship because of it. Without telling his parents, he left the college, eventually ending up in Madras, where he took the university entrance exam, but only passed the mathematics portion, and couldn't get in.

With very little money or support, Ramanujan kept up his research, and as India was under direct British rule at the time, he sent examples of his work to several English mathematicians. Following the example of Carr's book, he did not include proofs. The well-regarded mathematician G.H. Hardy sent a letter that was quite encouraging about Ramanujan's discoveries — Hardy later wrote,

> A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true because, if they were not true, no one would have had the imagination to invent them.*

Ramanujan replied to Hardy's letter, "I am already a half starving man. To preserve my brains I want food and this is my first consideration. Any sympathetic letter from you will be helpful to me here. . . ." Before long, Hardy had arranged for Ramanujan to move to England. Cambridge awarded him a bachelor's degree, later upgraded to a doctorate, on the basis of his research. Hardy's collaborator, J.E. Littlewood, helped to teach Ramanujan some of the things he would have learned in college, but "it was extremely difficult because every time some matter, which it was thought that Ramanujan needed to know, was mentioned, Ramanujan's response was an avalanche of original ideas. . . ."†

Ramanujan proved stunning new identities, especially about the partition function, but sadly his health continued to fail. His time in England was largely spent in nursing homes being treated for tuberculosis, a misdiagnosis. He returned to India in 1919, hoping the familiar climate and diet would help, but died in 1920 at the age of 32, likely from a liver infection. The notebooks he left behind are still a fruitful source of new ideas.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Ramanujan/`

*G.H. Hardy, *Ramanujan: twelve lectures on subjects suggested by his life and work,* Cambridge University Press (1940), 9.

†E Shils, "Reflections on tradition, centre and periphery and the universal validity of science : the significance of the life of S Ramanujan," *Minerva* **29** (1991), 393-419.

Alfréd Rényi

Alfréd Rényi was an excellent student whose interest in astronomy led him to mathematics. His college studies had to be postponed, though, because in 1939, Budapest University imposed a limit on the number of Jewish students. He managed to enroll a year later, after working in a shipyard during his "gap year." Once he graduated, Hungary's pro-Nazi government conscripted him into a forced-labor battalion, but he managed to escape. By getting fake identification papers, he was able to stay in Budapest, and even saved his parents by impersonating a soldier and removing them from the ghetto in which they had been confined.

Despite the obvious difficulties, Rényi managed to earn his doctorate during World War II. Once the war ended, he spent a year in Russia, where he made important progress on two major problems in number theory: the Goldbach conjecture and the twin prime conjecture.

Back in Hungary, he began to focus on probability theory and its applications across all fields of math. He wrote the first probability theory text in the Hungarian language, starting from his own new set of axioms. With the famously eccentric and productive mathematician Paul Erdös and others, he blended probability with number theory and combinatorics. For example, he defined the Erdös-Rényi model of random graphs, and used it to give clever proofs that certain elusive types of graph do actually exist, by finding a positive probability that a randomly generated graph is of the desired kind.

Rényi remembered that math is fun. He described his profession in a memorable way:

> A mathematician is a machine for turning coffee into theorems.

[His teacher and friend Paul Turán, who had opinions, claimed that American coffee was only good for lemmas.] But in a more serious vein, he also wrote the *Dialogues on Mathematics*, using Socrates, Archimedes, and Galileo as characters, to help the public understand what math is, and to share the joy he found in working as a mathematician.

Rényi summed up his approach to his work in one simple remark:

> If I feel unhappy, I do mathematics to become happy. If I am happy, I do mathematics to keep happy.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Renyi/`

Georg Friedrich Bernhard Riemann studied math with Dirichlet and others at Berlin University, then earned his doctorate at the University of Göttingen, with Gauss as his advisor. In his Ph.D. thesis, Riemann studied the properties of complex functions, and introduced fundamental ideas in complex analysis, including what we now call the Riemann mapping theorem, the Cauchy-Riemann equations (though these were known much earlier), and Riemann surfaces.

Bernhard Riemann

In the German system, Riemann now had to write a second thesis (his "Habilitation") to earn a job as an independent university lecturer. Gauss assigned the topic: the foundations of geometry. Riemann, who had already made major discoveries in a different field, was stunned and discouraged, but eventually succeeded in writing the paper ("On the hypotheses which underlie geometry"), and it was revolutionary: he invented the "Riemannian manifold" and the concept of the curvature of space, which would turn out to be exactly what Einstein needed for his general theory of relativity.

Calculus students learn about Riemann when studying the Riemann sum:

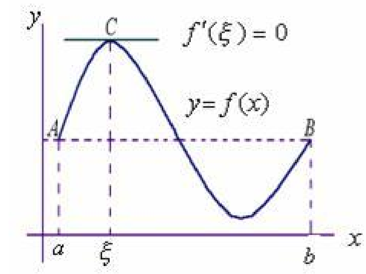
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

This is how we define the definite integral now, but integrals were around long before Riemann. Cauchy, in 1823, was the first person to give a formal definition of definite integrals for continuous functions. But to work with Fourier series, it was important to be able to integrate non-continuous functions too, and as part of his habilitation thesis, Riemann extended Cauchy's work to apply in that context.

The prime-counting function π, for which $\pi(x)$ represents the number of primes less than or equal to x, was an important object of study at the time (and still is). While seeking a formula for $\pi(x)$, Riemann had to work with the function $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$, which we now call the Riemann zeta function. He conjectured that if $\zeta(z) = 0$, then either z is an even negative integer or $z = \frac{1}{2} + iy$ for some real number y. This claim is the "Riemann hypothesis," and it still hasn't been proved or disproved: doing so is one of the Clay Mathematics Institute's Millennium Problems, with a million-dollar prize attached.

Sadly, Riemann caught tuberculosis while still quite a young man, and died before his 40^{th} birthday.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Riemann/`



Michel Rolle

Michel Rolle didn't have the opportunity to go to school past the elementary level, but studied math on his own, and did well enough that as a young man living in Paris, he solved a challenge problem published by a successful textbook author. Powerful people noticed: he got hired as a private tutor for the son of the Marquis de Louvois, who was France's Secretary of State for War. In just a few years, Louvois got Rolle named to the Royal Academy of Sciences, so he was able to discuss mathematics with the leading experts in Paris.

In 1690, Rolle published the book *Traité d'algèbre ("Treatise on algebra")*, in which the radical notation $\sqrt[n]{x}$ for n^{th} roots appears for the first time. The next year, he put out another work in which he justified some of the methods he had used in the *Traité* without proving their validity. Here he shows that if $f(a) = f(b)$ with $a \neq b$, then there is some c between a and b where $f'(c) = 0$. It took 150 years before someone called this "Rolle's theorem" for the first time. In the same work, Rolle started using the "$=$" sign to represent equality: this was invented in 1557 (!) by Robert Recorde in England, but it still wasn't common in Rolle's time. Finally, Rolle committed to another rule that now seems very natural: When you compare two negative numbers, the one closer to zero is greater than the other. This, too, went against the standard practice that had been established by Descartes.

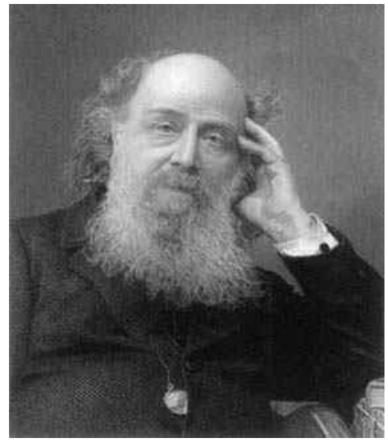
Although he is remembered for a theorem in differential calculus, Rolle was strongly critical of that subject. He argued in the Academy of Sciences that it was wrong to treat differentials as infinitesimal changes in variables:

> [E]xactness does not reign anymore in geometry since the new system of infinitely small quantities has been mixed to it. I do not see that this system has produced anything for the truth and it would seem to me that it often conceals mistakes.

At that time (1700), the Academy could not determine whether Rolle was right about this or not, and the arguments were so fierce that they banned further debate on the subject. In the end, Rolle conceded that he had been wrong, but by modern standards, maybe he wasn't! As Rolle demanded, we now do calculus in the context of the real numbers, where there are no infinitely small quantities. The definitions that make this possible were only established by Cauchy and Weierstrass in the 1800s.

At the age of 56, Rolle suffered a stroke, which put an end to his research, though he lived another 11 years before dying of a second stroke. As far as we know, there aren't any pictures of Rolle; the one that you see online most often is actually a portrait of Leibniz.

Image: Fs448445223 via Wikimedia Commons, `https://tinyurl.com/rollesthm`.



J. J. Sylvester

James Joseph was born in England in 1814. He only took the last name Sylvester as a teenager, when his brother found out that he couldn't move to the United States without having a first, middle, and last name. He started college at fourteen, in the brand-new University College London, with the famous logician Augustus De Morgan for a teacher. However, he didn't last a full year there, leaving in a hurry after he was accused of threatening another student with a knife in the cafeteria.

A few years older and wiser, he enrolled at Cambridge University, where he was a student for six years — he was sick for quite a while and wasn't able to be at school for about two years. Though he was an excellent math student, he did not graduate from Cambridge because he was Jewish, and was not willing to swear that he believed in the doctrine of the Church of England. Despite his lack of a degree, he became a professor of natural philosophy (physics) at the University of London, where De Morgan was also working.

Wanting to find a job teaching mathematics, Sylvester left England to serve as a professor at the University of Virginia. His colleague William Rogers wrote that Sylvester "was terribly embarrassed at his first lecture, indeed quite overwhelmed, but has been doing better since. He has a good deal of hesitation, is not fluent, but is very enthusiastic...." However, he met with antisemitic protests, as well as accusations of "ignorance of our peculiar Institutions" (that is, slavery). He and other faculty had to fear actual violence from the students. Eventually, when Sylvester objected to a student's misbehavior (reading a newspaper in class), he and his brother demanded an apology, and hit Sylvester in the head with a club. Sylvester, however, was carrying a sword-cane, and he drew the blade and stabbed the student. Not waiting to learn that he had not seriously hurt the man, he abandoned his job and returned to England.

Back in London, Sylvester worked as an actuary and as a private math tutor (with Florence Nightingale among his students), and studied law, where he met the great algebraist Arthur Cayley. He began to do his own important work in linear algebra, and became a professor at the Royal Military Academy. Forced to retire at age 55, he eventually moved back to the United States, becoming a professor at Johns Hopkins in Baltimore. During his seven years there, he started the first journal of mathematics in the U.S. He then spent ten years as a professor at Oxford before finally retiring to London.

Sylvester's greatest impact was probably on the vocabulary of mathematics: he gave us such terms as matrix, discriminant, isomorphism, and many others. Referring to the Biblical story of the naming of the animals, Sylvester wrote, "Perhaps I may without immodesty lay claim to the appellation of Mathematical Adam, as I believe that I have given more names... than all the other mathematicians of the age combined."
Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Sylvester/`

Brook Taylor

Besides his interest in mathematics, Brook Taylor had a good education in painting and music. His well-off family (his father was a member of Parliament, and his mother's father was a baronet) hired private tutors for him, and he would later do original mathematical work related to both. In his charmingly-titled 1715 book *Linear Perspective: Or, a New Method of Representing Justly All Manner of Objects as They Appear to the Eye in All Situations*, he was the first to thoroughly study vanishing points in perspective drawing.

After graduating from Cambridge in 1709, Taylor worked to refine the ideas he'd developed in college. He was elected to the prestigious Royal Society in 1714, and served for four years as their Secretary.

In 1715, Taylor published the book *Methodus incrementorum directa et inversa*, in which he introduced the "Taylor series" for which he is best remembered. Though it's named after him, Taylor was not the first to work with Taylor series; they were already used by both Newton and Leibniz, among others. Taylor rediscovered the series through his work related to Kepler's problem on planetary motion. He writes in his book that he was inspired by a conversation he had in a coffeehouse with his former tutor John Machin. Joseph-Louis Lagrange, nearly sixty years later, was the first to single out Taylor series as a fundamental idea in calculus.

Though the introduction of Taylor series would have been enough to make the book memorable, Taylor also invented the method of integration by parts in *Methodus incrementorum directa et inversa*. The same book also contains the first study of what we now call the "calculus of finite differences."

Taylor was deeply involved in the bitter priority disputes between mathematicians in England and continental Europe. He was part of the English committee formed to determine whether Leibniz or Newton was responsible for the discovery of calculus. (The committee gave credit to Newton, but since all the members were English and most were Newton's personal friends, their judgment wasn't taken very seriously in France or Germany.) In a series of public letters, he exchanged strong words and mathematical challenges with Johann Bernoulli and others.

Taylor married against his parents' will, causing a break in their relationship. Sadly, his wife died in childbirth, and Taylor moved back in with his parents. They reconciled well enough that Taylor inherited his parents' estate. He remarried, and his second wife also died giving birth to a daughter. Taylor himself died the following year, of unknown causes, at the age of 46.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Taylor/`

At the age of eight, Valerie Thomas went to the library and checked out a book called *The Boy's First Book on Electronics*. She took it home to her father, who was interested in taking apart radios and other home electronic devices, but she couldn't get him to work on the projects with her: he didn't think it was a good hobby for girls. When she was a little older, she went to an all-girls high school, which had only recently been racially integrated. The school was supportive in some ways, but didn't offer her the opportunity to participate in advanced classes in math and science.

Valerie Thomas

It was when Thomas went to college at Morgan State University, a historically black university in Baltimore, that she finally received real encouragement. She graduated with honors, majoring in physics, and took a job at NASA working as a data analyst. She said that when she got there in 1964,

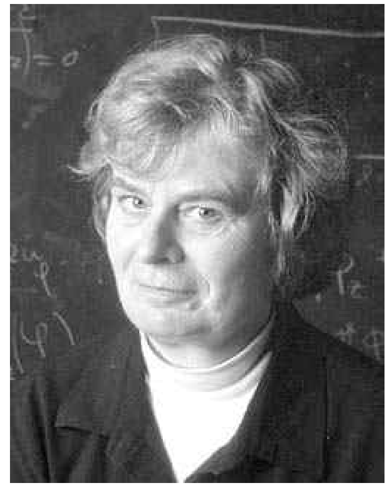
> I had not seen a computer except in science fiction movies. Since my job involved writing computer programs, I decided to learn as much as possible about computers.

NASA assigned her to work with Landsat, the first family of satellites that could take pictures of Earth from space. Thomas wrote the computer code (on key punch cards, fed into a shared office computer) that translated the raw data from the satellite into visual images that people could use, including false-color images that displayed properties like temperature or vegetation. She then leveraged that data to make it possible, for the first time, to use satellite imagery to predict wheat harvest data around the world.

Between 1976 and 1978, Thomas invented and patented a tool called the "illusion transmitter," which uses concave parabolic mirrors to send three-dimensional images (in the form of stereographic optical illusions) in real time across long distances, and which is still used by NASA today.

Thomas also worked in the Space Physics Analysis Network. There she worked on the Voyager spacecraft project, among others. By the time she retired in 1995, Thomas was associate chief of NASA's Space Science Data Operations Office, and received the Goddard Space Flight Center Award of Merit. She has been an active mentor for many students, most recently through organizations like Shades of Blue, which focuses on helping young people to pursue careers in aviation and aerospace. Since her retirement, she has worked to inspire students by serving as a substitute teacher in a Baltimore-area high school.

Image: NASA, `https://en.wikipedia.org/wiki/Valerie_Thomas`



Karen Uhlenbeck

Born in Cleveland, Ohio, Karen Uhlenbeck (*née* Keskulla) grew up in New Jersey, a curious girl who loved reading, art and music, and the outdoors. She went to college at the University of Michigan, and then went to grad school at New York University. While there, she married Olke Uhlenbeck, a biophysicist who later did important research on RNA. When he started his doctoral program at Harvard University, she went along, meaning she had to restart her graduate work at a new school — Brandeis University.

After earning the Ph.D. in 1968, she had trouble finding a suitable job as a professor: many schools were simply not willing to consider hiring a woman. She joined the faculty of the University of Illinois, first at the main campus and then at UI-Chicago. Her excellent work on partial differential equations earned her a MacArthur Fellowship, a significant honor, which comes with a major, no-strings-attached cash prize, and is commonly referred to simply as "the genius grant." Since that time, Uhlenbeck has worked at the University of Chicago, the University of Texas at Austin, Princeton, and the Institute for Advanced Study.

Uhlenbeck helped to create the field of geometric analysis, in which differential equations are used to study the properties of geometric (or topological) spaces. Uhlenbeck made it possible to use "instantons," which are non-trivial solutions to the equations describing how fields (like electromagnetism and gravitation) interact with matter. These partial differential equations, the "self-dual Yang-Mills equations" and other "Yang-Mills theories," are foundational to what we now call the Standard Model of elementary particle physics. Uhlenbeck's classic 1984 textbook, Instantons and 4-Manifolds, written jointly with Dan Freed, includes her famous "removable singularities theorem" as an appendix. This book, and Uhlenbeck's work around it, helped to shape research in mathematical physics for decades to come.

Uhlenbeck has won a dizzying array of awards, culminating, in 2019, with the Abel Prize. Named for the great 19th-century Norwegian mathematician Niels Henrik Abel, this award is presented by the King of Norway, and carries a prize of 7.5 million Norwegian kroner (a little over $700,000). Along with the Fields Medal (which is awarded only to mathematicians under the age of 40), the Abel Prize is regarded as the highest honor in mathematics, comparable to a Nobel Prize. The award citation honors "her pioneering achievements in geometric partial differential equations, gauge theory and integrable systems, and... the fundamental impact of her work on analysis, geometry and mathematical physics," and adds that "her ideas and leadership have transformed the mathematical landscape as a whole."

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Uhlenbeck/`

Karl Weierstrass wanted to be a mathematician — he was already reading mathematical research journals in high school — but his father insisted that he study finance in college. At the university, forbidden to pursue a math degree and indifferent to economics, he spent his time drinking and fencing, then dropped out of school after four years. Furious, his father sent him to a new school, where he could study to become a math teacher. He went willingly, because there he'd also have the opportunity to learn from a professional mathematician.

Karl Weierstrass

Weierstrass did finish this program, and spent the next decade teaching high school. While there, he didn't have other mathematicians to talk to, and he wasn't in good health. Still, he spent all of his free time thinking about mathematics, and it paid off. As he turned 40, he published a pair of major papers in an extremely influential journal that circulated all across Europe. The results were everything he could have asked for: suddenly, he was a superstar. He was awarded an honorary doctorate, and started to get job offers. Eventually he became a professor at the University of Berlin, and many of his students there (including Kovalevskaya, Mittag-Leffler, and Cantor) ended up achieving great things.

Weierstrass devoted most of his research to the study of analytic functions: complex functions that, near any given point z_0 in the domain, can be written in the form of a power series $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$. Normally the domain of such a series is only a disk in the complex plane, but Weierstrass showed how to join these disks together to build up the "analytic continuation" of $f(z)$. On the real side of things, he showed that any function with domain $[a, b]$ can be approximated as closely as you want by a polynomial — great news, since evaluating a polynomial just involves multiplying and adding.

It was Weierstrass who first gave (in 1861) the modern definition of the limit in terms of real numbers, and he invented the familiar $\lim_{x \to a}$ notation. He was then able to define, for the first time, what it means for a function $f(x)$ to be continuous at a. Using these definitions, he achieved a goal set earlier by Riemann: he found a family of specific functions, like $f(x) = \sum_{n=1}^{\infty} \frac{\cos 4^n x}{2^n}$, which were continuous everywhere, but were not differentiable anywhere! (Even more astonishing, Banach would later prove that almost all continuous functions are nowhere differentiable: it turns out that the smooth functions that we usually study are really the outliers.) This shocking pair of properties motivated a lot of new work, forcing mathematicians to reach beyond their intuitive ideas about functions and recognize a larger, stranger world of possibilities.

Image: MacTutor, `https://mathshistory.st-andrews.ac.uk/Biographies/Weierstrass/`